The Method of Stationary Phase

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Oscillatory integrals of the first kind

Given $n \in \mathbb{N}$ we will study

$$I_{u,\varphi}(\lambda) = \int_{\mathbb{R}^n} u(x)\, \mathrm{e}^{i\lambda\varphi(x)}\, \mathrm{d}x$$

for $u \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ and $\lambda \in \mathbb{R}$.

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Example

When $n = 1$ and $\varphi = -\mathrm{id}$ we have

$$I_{u,-\mathrm{id}}(\lambda) = \int_{-\infty}^{\infty} u(x)\, \mathrm{e}^{-i\lambda x}\, \mathrm{d}x = \mathscr{F}u(\lambda).$$

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Riemann-Lebesgue lemma: $I_{u,-\mathrm{id}}(\lambda) \to 0$ as $\lambda \to \pm\infty$.

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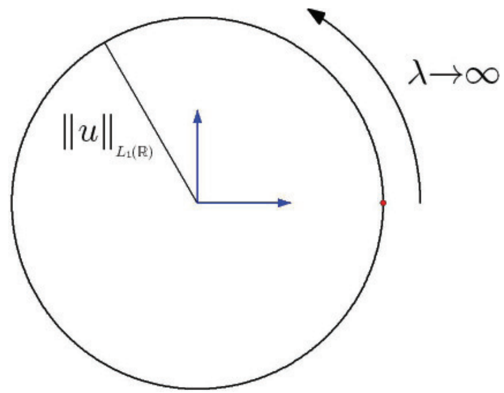
Example

Setting $n = 1$, $u > 0$ and $\varphi = 1$ gives

$$I_{u,1}(\lambda) = \int_{\mathbb{R}^n} u(x) \mathrm{e}^{i\lambda}\, \mathrm{d}x = \mathrm{e}^{i\lambda} \|u\|_{L_1(\mathbb{R})}$$



Principle of non-stationary phase

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Proof: On the blackboard (see exercise 3.1).

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Assumption: The stationary points $y \in \operatorname{supp}(u)$ of φ are non-degenerate (i.e. $\det(\partial_i\partial_j\varphi(y))_{ij} \neq 0$).

The Morse Lemma

Lemma

Let $x_0 \in \mathbb{R}^n$ be a non-degenerate stationary point of $\varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$. Then there exist neighbourhoods V of x_0 and U of $0 \in \mathbb{R}^n$, numbers $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ and a diffeomorphism $\mathcal{H} : V \to U$ with $\mathcal{H}(x_0) = 0$ such that

$$\varphi \circ \mathcal{H}^{-1}(x) = \varphi(x_0) + \varepsilon_1 x_1^2 + \cdots + \varepsilon_n x_n^2$$

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with $\mathcal{E} = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$.

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with $\mathcal{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$.

Remark: It can be shown that the number of $+1$'s amongst $\varepsilon_1, \ldots, \varepsilon_n$ is equal to the number of positive eigenvalues of $(\partial_i \partial_j \varphi(x_0))_{ij}$

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Without loss of generality assume that $x_0 = 0$ and $\varphi(0) = 0$.

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After proving the case with $x_0 = 0$ and $\varphi(0) = 0$, apply the obtained result to the function $x \mapsto (\varphi(x + x_0) - \varphi(x_0))$.

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On the blackboard we will show the following statement:

Proof of the Morse Lemma

Without loss of generality assume that $x_0 = 0$ and $\varphi(0) = 0$.

For all $N \in \{1, \ldots, n+1\}$ there exist neighbourhoods $V_N, U_N \subset \mathbb{R}^n$ of 0, a diffeomorphism $\mathcal{H}_N : V_N \to U_N$ with $\mathcal{H}_N(0) = 0$, numbers $\varepsilon_m \in \{\pm 1\}$ and a set of functions $\{q_{ij}^{(N)} \mid i, j \in \mathbb{N}, N \le i, j \le n\}$ with

(i_N) $q_{ij}^{(N)} \in C^\infty(V_N)$,

(ii_N) $q_{ij}^{(N)} = q_{ji}^{(N)}$,

(iii_N) $q_{\ell k}^{(N)}(0) \neq 0$ for some ℓ, k

such that

$$\varphi \circ \mathcal{H}_N^{-1}(x) = \sum_{m=1}^{N-1} \varepsilon_m x_m^2 + \sum_{N \le i,j \le n} q_{ij}^{(N)}(x) x_i x_j.$$

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$$\varphi \circ \mathcal{H}_N^{-1}(x) = \underbrace{\sum_{m=1}^{N-1} \varepsilon_m x_m^2}_{=0 \text{ if } N=1} + \sum_{N \leq i,j \leq n} q_{ij}^{(N)}(x) x_i x_j.$$

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$= 0$ if $N = n+1$

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Consequences of the Morse Lemma

Corollary

A non-degenerate stationary point x_0 of $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ is an isolated stationary point.

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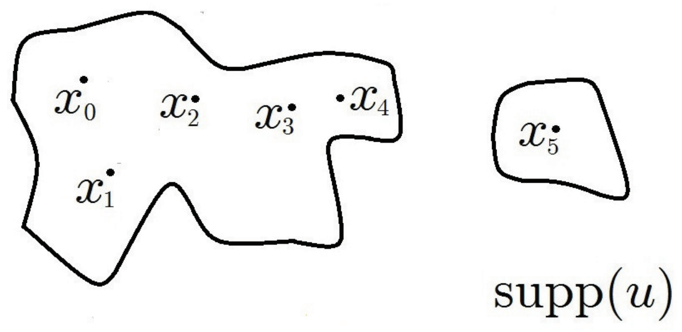
The compact set $\operatorname{supp}(u)$ can only contain finitely many non-degenerate stationary points of φ.



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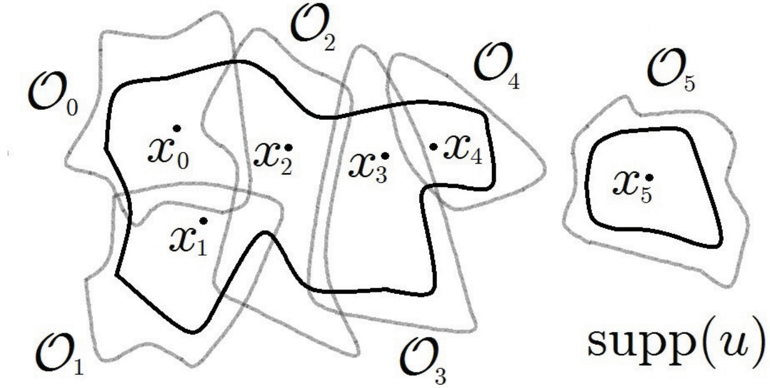
A non-degenerate stationary point x_0 of $\varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is an isolated stationary point.

Let $\{\mathcal{O}_j\}_{j=0}^N$ be a bounded open cover of $\operatorname{supp}(u)$ such that $\mathcal{O}_j$ contains precisely one stationary point of φ.



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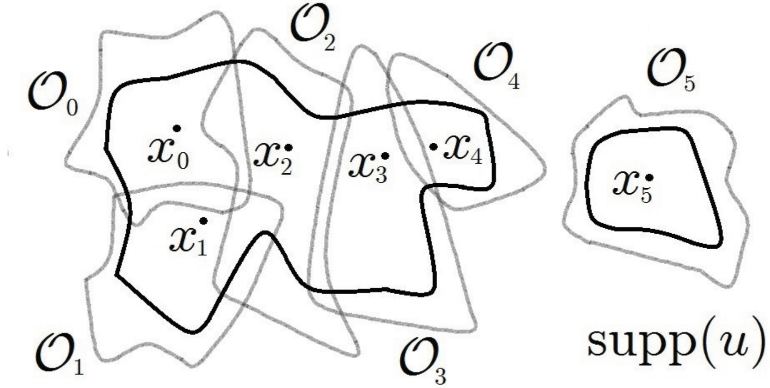

Partition of unity: $\sum_{j=0}^N \psi_j = 1$ on $\operatorname{supp}(u)$ and $\psi_j \in C_c^\infty(\mathcal{O}_j;[0,1])$.

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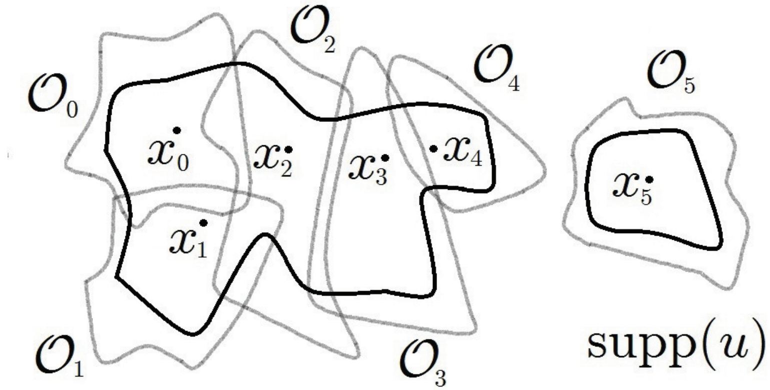

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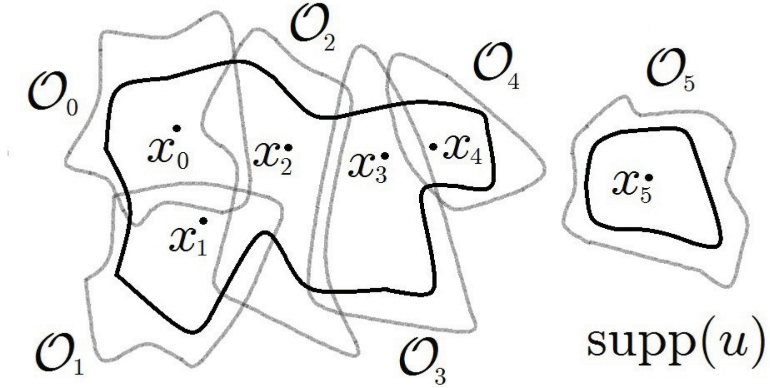

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Can assume: φ has precisely one stationary point in $\operatorname{supp}(u)$.

Special Case: Quadratic Forms

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Proposition

Let A be a real, symmetric and invertible $n \times n$-matrix. Then for all $u \in C_c^\infty(\mathbb{R}^n)$, $\lambda > 0$ and all integers $k > 0$ and $s > \frac{n}{2}$ we have

$$\left| I_{u,\langle \cdot, A \cdot \rangle}(\lambda) - \left(\det\left(\frac{A}{\pi i} \right) \right)^{-\frac{1}{2}} \sum_{j=0}^{k-1} \frac{\langle D, A^{-1} D \rangle^j u(0)}{(4i)^j j!} \lambda^{-\frac{n}{2}-j} \right|$$

$$\leq C_k \left(\frac{\|A^{-1}\|}{\lambda} \right)^{\frac{n}{2}+k} \sum_{|\alpha| \leq s+2k} \|D^\alpha u\|_{L^2},$$

where $D = \frac{1}{i}(\partial_1, \ldots, \partial_n)$.

Proof: On the blackboard

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Theorem

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where T_j is a differential operator of order $2j$ with C^∞-coefficients.

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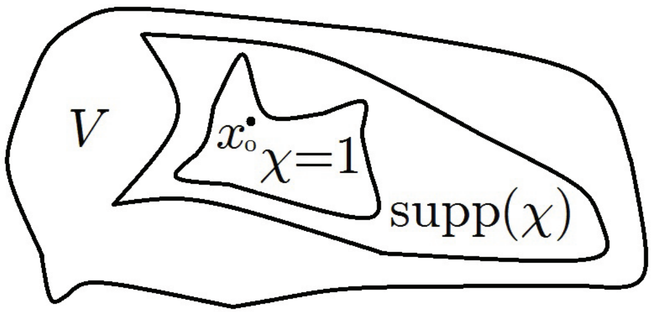
Choose $\chi \in C_c^\infty(V)$ with $\chi = 1$ near x_0.



Principle of Stationary Phase (proof)

Then

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\begin{aligned}
&I_{u,\varphi}(\lambda) \\
&= \int_{\mathbb{R}^n} \mathrm{e}^{i\lambda\varphi(x)} (\chi u)(x)\,\mathrm{d}x + \int_{\mathbb{R}^n} \mathrm{e}^{i\lambda\varphi(x)} [(1-\chi)u](x)\,\mathrm{d}x
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&= \int_U \mathrm{e}^{i\lambda\varphi\circ\mathcal{H}^{-1}(x)} \qquad (\chi u)\circ\mathcal{H}^{-1}(x)\,|\mathrm{det}J\mathcal{H}^{-1}(x)|\,\mathrm{d}x + I_{(1-\chi)u,\varphi}(\lambda)
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\begin{aligned}
&I_{u,\varphi}(\lambda)\\
&= \int_V e^{i\lambda\varphi(x)}(\chi u)(x)\,\mathrm{d}x + \int_{\mathbb{R}^n} e^{i\lambda\varphi(x)}[(1-\chi)u](x)\,\mathrm{d}x\\
&= \int_U e^{i\lambda\left(\varphi(x_0)\,+\,\langle x,\,\mathcal{E}x\rangle\right)} \qquad f_u(x) \qquad \mathrm{d}x + I_{(1-\chi)u,\varphi}(\lambda)\\
&= e^{i\lambda\varphi(x_0)} I_{f_u,\langle\cdot,\mathcal{E}\cdot\rangle}(\lambda) + I_{(1-\chi)u,\varphi}(\lambda)
\end{aligned}
$$

Principle of Stationary Phase (proof)

Then

$$I_{u,\varphi}(\lambda) = \mathrm{e}^{i\lambda\varphi(x_0)} I_{f_u,\langle\cdot,\mathcal{E}\cdot\rangle}(\lambda) + I_{(1-\chi)u,\varphi}(\lambda)$$

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so by setting $T_j u = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} \frac{\langle D,\mathcal{E}^{-1}D\rangle^j f_u}{(4i)^j j!}$ and letting s be the smallest integer $> \frac{n}{2}$ we get

$$\left| I_{u,\varphi}(\lambda) - \mathrm{e}^{i\lambda\varphi(x_0)} \sum_{j=0}^{k-1} T_j u(0) \lambda^{-\frac{n}{2}-j} \right|$$

$$\leq \left| I_{f_u,\langle\cdot,\mathcal{E}\cdot\rangle}(\lambda) - \sum_{j=0}^{k-1} T_j u(0) \lambda^{-\frac{n}{2}-j} \right| + \left| I_{(1-\chi)u,\varphi}(\lambda) \right|$$

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$$\leq C_k \|\mathcal{E}^{-1}\|^{\frac{n}{2}+k} \sum_{|\alpha|\leq 2k+s} \|D^\alpha f_u\|_{L^2} \lambda^{-\frac{n}{2}-k} + C'_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-k}$$

Principle of Stationary Phase (proof)

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$$\leq C_{k,n,u,\varphi}\lambda^{-\frac{n}{2}-k}$$

The simplest asymptotic expansion of $I_{u,\varphi}(\lambda)$

Remembering the definitions of $T_j u$ and f_u,

$$T_j u = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} \frac{\langle D, \mathcal{E}^{-1} D\rangle^j f_u}{(4i)^j j!}$$

and

$$f_u = \left[(\chi u) \circ \mathcal{H}^{-1}\right] \cdot \left|\det J\mathcal{H}^{-1}\right|,$$

we see that

$$T_0 u(0) = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} f_u(0) = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} \left|\det J\mathcal{H}^{-1}(0)\right| u(x_0)$$

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$$T_0 u(0) = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} f_u(0) = \qquad C_{(\partial_i \partial_j \varphi(x_0))_{ij}} \qquad u(x_0)$$

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so

$$\left| I_{u,\varphi}(\lambda) - C_{(\partial_i \partial_j \varphi(x_0))_{ij}}\, \mathrm{e}^{i\lambda\varphi(x_0)} u(x_0) \lambda^{-\frac{n}{2}} \right| \leq C_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-1}.$$

Final remarks

Topics for further studies

- Considering $I_{u,\varphi}(\lambda)$ with complex λ or complex ϕ,

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- Considering $I_{u,\varphi}(\lambda)$ with complex λ or complex ϕ,
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KIM PETERSEN

Stationary Phase

Given $n\in\mathbb{N}$ we will study

$$I_{u,\varphi}(\lambda)=\int_{\mathbb{R}^n}\overbrace{u(x)}^{\text{amplitude}}e^{i\lambda\overbrace{\varphi(x)}^{\text{phase}}}\,dx=\int_{\mathbb{R}^n}ue^{i\lambda\varphi}\,dm$$

for $u\in C_c^\infty(\mathbb{R}^n)$, $\varphi\in C^\infty(\mathbb{R}^n;\mathbb{R})$ and $\lambda\in\mathbb{R}$.

Example 1

When $n=1$ and $\varphi=-id$ we have

$$I_{u,-id}(\lambda)=\int_{-\infty}^{\infty}u(x)e^{-i\lambda x}\,dx=\mathcal{F}u(\lambda).$$

Riemann-Lebesgue lemma: $I_{u,-id}(\lambda)\longrightarrow 0$ as $\lambda\to\pm\infty$ (even for $u\in\mathcal{L}^1(\mathbb{R})$)

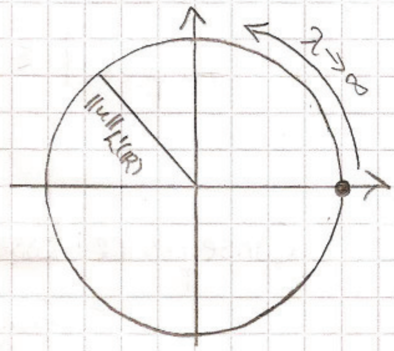
How does $I_{u,\varphi}(\lambda)$ behave as $\lambda\to\underset{(-)}{\pm}\infty$ for general $\varphi\in C_c^\infty(\mathbb{R}^n;\mathbb{R})$?

$$\overline{I_{\bar u,\varphi}(\lambda)}=\overline{\int_{\mathbb{R}^n}\bar u e^{i\lambda\varphi}dm}=\int_{\mathbb{R}^n}ue^{-i\lambda\varphi}dm=I_{u,\varphi}(-\lambda)$$

Example 2:

When $n=1$, $u>0$ and $\varphi=1$ we have

$$I_{u,1}(\lambda)=e^{i\lambda}\|u\|_{L^1(\mathbb{R})}$$



"Complicated behavior"

Principle of non-stationary phase (see exercise 3.1)

Let $u\in C_c^\infty(\mathbb{R}^n)$ and let $\varphi\in C^\infty(\mathbb{R}^n;\mathbb{R})$ such that $\nabla\varphi$ is non-zero on supp$u$ (e.g. as in example 1). Then

$$|I_{u,\varphi}(\lambda)|\le C_{N,u,\varphi}\,\lambda^{-N}\quad\text{for all } N\in\mathbb{N}_0 \text{ and } \lambda>0$$

Proof: Note that on supp$u$ we have

$$\frac{1}{i\lambda}\frac{\nabla\varphi}{|\nabla\varphi|^2}\cdot\nabla(e^{i\lambda\varphi})=\cancel{\frac{1}{i\lambda}}\frac{\cancel{\nabla\varphi}}{\cancel{|\nabla\varphi|^2}}\cdot(e^{i\lambda\varphi}\cdot\cancel{i\lambda\nabla\varphi})=e^{i\lambda\varphi}$$

non-zero!

Stationary Phase

so

$$I_{u,\varphi}(\lambda) = \frac{1}{i\lambda}\int_{\mathbb{R}^n} u\,\frac{\nabla\varphi}{|\nabla\varphi|^2}\cdot\nabla(e^{i\lambda\varphi})\,dm$$

$$= -\frac{1}{i\lambda}\int_{\mathbb{R}^n} \underbrace{\nabla\cdot\left(u\,\frac{\nabla\varphi}{|\nabla\varphi|^2}\right)}_{=u_1\in C_c^\infty(\mathbb{R}^n)\text{ w/ }\operatorname{supp}u_1\subset\operatorname{supp}u,\text{ dep. only on }u,\varphi} e^{i\lambda\varphi}\,dm$$

$$= -\frac{1}{i\lambda}\, I_{u_1,\varphi}(\lambda)$$

$$= \left(-\frac{1}{i\lambda}\right)^2 I_{u_2,\varphi}(\lambda) \qquad u_2 = \nabla\cdot\left(u_1\frac{\nabla\varphi}{|\nabla\varphi|^2}\right)$$

$$\vdots$$

$$= \left(-\frac{1}{i\lambda}\right)^N I_{u_N,\varphi}(\lambda) \qquad u_N = \nabla\cdot\left(u_{N-1}\frac{\nabla\varphi}{|\nabla\varphi|^2}\right)$$

Hence

$$|I_{u,\varphi}(\lambda)| \le \lambda^{-N}\underbrace{\int_{\operatorname{supp}u}|u_N(x)|\,dx}_{=C_{N,u,\varphi}}$$

□

Consequence: Essential contributions to the asymptotic behavior of $I_{u,\varphi}$ come from the stationary points of φ (i.e. points $y\in\mathbb{R}^n$ with $\nabla\varphi(y)=0$)

General assumption: The stationary points $y\in\operatorname{supp}u$ of φ are non-degenerate (i.e. $\det(\partial_i\partial_j\varphi(y))\neq 0$).

The Morse Lemma: Let $x_0\in\mathbb{R}^n$ be a non-degenerate stationary point of $\varphi\in C^\infty(\mathbb{R}^n;\mathbb{R})$. Then there are ngbh's V of x_0 and $\mathcal{U}$ of $0\in\mathbb{R}^n$, numbers $\varepsilon_1,\dots,\varepsilon_n\in\{\pm 1\}$ and a diffeomorphism $\mathcal{H}: V\to\mathcal{U}$ with $\mathcal{H}(x_0)=0$ such that

$$\varphi\circ\mathcal{H}^{-1}(x) = \varphi(x_0)+\varepsilon_1 x_1^2+\dots+\varepsilon_n x_n^2 = \varphi(x_0)+\langle x,\mathcal{E}x\rangle \quad\text{with } \mathcal{E}=\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n\end{pmatrix}$$

Remark: It can be shown that the number of $+1$'s amongst $\varepsilon_1,\dots,\varepsilon_n$ is equal to the number of positive eigenvalues of $(\partial_i\partial_j\varphi(x_0))_{ij}$.

Stationary Phase

Corollary

A non-degenerate stationary point x_0 of $\varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is an isolated stationary point.

Proof:

For $x \in \mathcal{U}$

$$\underbrace{[J\mathcal{H}^{-1}(x)]}_{\text{Jacobian}}{}^{\overset{\text{invertible (diffeom)}}{T}} \nabla\varphi(\mathcal{H}^{-1}(x)) \overset{\text{chainrule}}{=} \nabla(\varphi\circ\mathcal{H}^{-1})(x) = 2\begin{pmatrix}\varepsilon_1 x_1\\ \vdots \\ \varepsilon_n x_n\end{pmatrix}$$

so setting $x = \mathcal{H}(y)$ gives

$$\nabla\varphi(y) = 2\left([J\mathcal{H}^{-1}(\mathcal{H}(y))]^T\right)^{-1}\begin{pmatrix}\varepsilon_1[\mathcal{H}(y)]_1\\ \vdots \\ \varepsilon_n[\mathcal{H}(y)]_n\end{pmatrix} \neq 0 \text{ for } y \in V\setminus\{x_0\}$$ ▨

Proof of Morse Lemma:

Wlog assume that $x_0 = 0$ and $\varphi(0) = 0$.

[After proving this case, apply the result to $x \mapsto (\varphi(x+x_0) - \varphi(x_0))$]

We will show:

(*) For all $N \in \{1,\dots,n+1\}$ there exist nbhd's $V_N, \mathcal{U}_N \subseteq \mathbb{R}^n$ of 0, a diffeomorphism $\mathcal{H}_N : V_N \to \mathcal{U}_N$ with $\mathcal{H}_N(0) = 0$, numbers $\varepsilon_m \in \{\pm 1\}$ and a set of functions $\{q_{ij}^{(N)} \mid i,j \in \mathbb{N},\, N \leq i,j \leq n\}$ with

i_N) $q_{ij}^{(N)} \in C^\infty(V_N)$,

ii_N) $q_{ij}^{(N)} = q_{ji}^{(N)}$,

iii_N) $q_{lk}^{(N)}(0) \neq 0$ for some l,k

such that

$$\varphi\circ\mathcal{H}_N^{-1}(x) = \underbrace{\sum_{m=1}^{N-1}\varepsilon_m x_m^2}_{=0 \text{ if } N=1} + \underbrace{\sum_{N\leq i,j\leq n} q_{ij}^{(N)}(x)\, x_i x_j}_{=0 \text{ if } N=n+1.}$$

Induction start (N = 1): By the Taylor formula [GG, (A.8)]

$$\varphi(x) = \sum_{|\alpha|<2}\frac{x^\alpha}{\alpha!}\partial^\alpha\varphi(0) + \sum_{|\alpha|=2}\frac{2}{\alpha!}x^\alpha\int_0^1(1-\theta)\partial^\alpha\varphi(\theta x)\,d\theta$$

$$= \sum_{1\leq i,j\leq n} q_{ij}^{(1)}(x)\, x_i x_j$$

Stationary Phase

with

$$q_{ij}^{(1)}(x) = \frac{2}{i!j!}\int_0^1 (1-\theta)\partial_i\partial_j\varphi(\theta x)\,d\theta.$$

We set $V_1 = U_1 = \mathbb{R}^n$, $\mathcal{H}_1 = id_{\mathbb{R}^n}$ and note that $q_{ij}^{(1)}$ satisfies $i_1) - iii_1)$

[$i_1)$ Trivial

$ii_1)$ Trivial

$iii_1)$ $q_{ij}^{(1)}(0) = \frac{2}{i!j!}\partial_i\partial_j\varphi(0)\left[\theta - \frac{1}{2}\theta^2\right]_0^1 = \frac{1}{i!j!}\partial_i\partial_j\varphi(0)$.
so the claim follows since 0 is a non-degenerate stationary point for φ.]

Induction step: Assume (*) holds for some $N \in \{1, \dots, n\}$. By $iii_N)$ we can wlog assume that $q_{NN}^{(N)}(0) \neq 0$ (with a suitable automorphism L on $\mathbb{R}^n$ one can write

$$\varphi \circ \mathcal{H}_N^{-1} \circ L(y) = \sum_{m=1}^{N-1} \varepsilon_m y_m^2 + \sum_{N \leq i,j \leq n} \tilde{q}_{ij}^{(N)}(y) y_i y_j$$

where $\tilde{q}_{ij}^{(N)}$ has the properties $i_N) - iii_N)$ and $\tilde{q}_{NN}^{(N)}(0) \neq 0$).

[If there exists a $p \in \{N, \dots, n\}$ with $q_{pp}^{(N)}(0) \neq 0$ we can choose $L: (x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ and $\tilde{q}_{ij}^{(N)} = q_{\sigma(i)\sigma(j)}^{(N)} \circ L$ where $\sigma: (1, \dots, N, \dots, p, \dots, n) \mapsto (1, \dots, p, \dots, N, \dots, n)$

If not, we divide into two cases:

If there exists an $r \in \{N, \dots, n\}$ such that $q_{rN}(0) \neq 0$ then we can choose $L: (x_1, \dots, x_r, \dots, x_n) \mapsto (x_1, \dots, x_N + x_r, \dots, x_n)$, $\tilde{q}_{jN}^{(N)} = \tilde{q}_{Nj}^{(N)} = (q_{Nj}^{(N)} + q_{rj}^{(N)}) \circ L$ for $j \neq N$, $\tilde{q}_{NN}^{(N)} = (q_{NN}^{(N)} + q_{rr}^{(N)} + 2q_{rN}^{(N)}) \circ L$ and $\tilde{q}_{ij}^{(N)} = q_{ij}^{(N)} \circ L$ otherwise.

If not, we choose $L: (x_1, \dots, x_\ell, \dots, x_k, \dots, x_n) \mapsto (x_1, \dots, x_\ell + x_N, \dots, x_k + x_N, \dots, x_n)$
$\tilde{q}_{jN}^{(N)} = \tilde{q}_{Nj}^{(N)} = (q_{Nj}^{(N)} + q_{\ell j}^{(N)} + q_{kj}^{(N)}) \circ L$ for $j \neq N$, $\tilde{q}_{NN}^{(N)} = (q_{NN}^{(N)} + 2q_{\ell N}^{(N)} + q_{\ell\ell}^{(N)} + 2q_{kN}^{(N)} + q_{kk}^{(N)} + 2q_{\ell k}^{(N)}) \circ L$
and $\tilde{q}_{ij}^{(N)} = q_{ij}^{(N)} \circ L$ otherwise.]

By continuity of $q_{NN}^{(N)}$ there exists a nghb. $W \subset V_N$ of 0 on which $q_{NN}^{(N)} \neq 0$.

Then with $\varepsilon_N = \operatorname{sign}(q_{NN}^{(N)}(0))$ we get

$$\varphi \circ \mathcal{H}^{-1}(x) = \sum_{m=1}^{N-1} \varepsilon_m x_m^2 + \sum_{N \le i,j \le n} q_{ij}^{(N)}(x) x_i x_j$$

$=0$ if $N=n$

$$= \sum_{m=1}^{N-1} \varepsilon_m x_m^2 + \varepsilon_N \Big(\underbrace{\sqrt{|q_{NN}^{(N)}(x)|}\, x_N + \frac{\varepsilon_N}{\sqrt{|q_{NN}^{(N)}(x)|}} \sum_{j=N+1}^{n} q_{Nj}^{(N)}(x) x_j}_{\ell_N(x)} \Big)^2 + \sum_{N+1 \le i,j \le n} \left(q_{ij}^{(N)}(x) - \frac{q_{Ni}^{(N)}(x) q_{Nj}^{(N)}(x)}{q_{NN}^{(N)}(x)} \right) x_i x_j$$

$$\Bigg[\varepsilon_N (\ell_N(x))^2 = \varepsilon_N \Big(|q_{NN}^{(N)}| x_N^2 + \frac{1}{|q_{NN}|} \Big(\sum_{j=N+1}^{n} q_{Nj}^{(N)} x_j \Big) \Big(\sum_{i=N+1}^{n} q_{Ni}^{(N)} x_i \Big) + 2\varepsilon_N \sum_{j=N+1}^{n} q_{Nj}^{(N)} x_N x_j \Big)$$

$$= \sum_{N+1 \le i,j \le n} \frac{q_{Ni}^{(N)} q_{Nj}^{(N)}}{q_{NN}^{(N)}} x_i x_j + \sum_{\substack{N \le i,j \le n \\ i=N \vee j=N}} q_{ij}^{(N)} x_i x_j \Bigg]$$

Note that ℓ_N is C^∞ on W with

$$\partial_k \ell_N(x) = \begin{cases} 0 & , 1 \le k < N \\ \dfrac{\varepsilon_N}{\sqrt{|q_{NN}^{(N)}(x)|}} \left(\sum_{j=N}^{n} \left(\partial_k q_{Nj}^{(N)}(x) - \dfrac{\partial_k q_{NN}^{(N)}(x) q_{Nj}^{(N)}(x)}{2 q_{NN}^{(N)}(x)} \right) x_j + q_{Nk}^{(N)}(x) \right) & , N \le k \le n \end{cases}$$

so defining $\mathcal{H}: W \to \mathbb{R}^n$ by

$$\mathcal{H}(x) = (x_1, \dots, x_{N-1}, \ell_N(x), x_{N+1}, \dots, x_n) = (y_1, \dots, y_{N-1}, y_N, y_{N+1}, \dots, y_n)$$

gives a C^∞-map with

$$\det J\mathcal{H}(0) = \begin{vmatrix} 1 & & & & & & \\ & \ddots & & & 0 & & \\ & & 1 & & & & \\ & & & \frac{\varepsilon_N q_{NN}^{(N)}(0)}{\sqrt{|q_{NN}^{(N)}(0)|}} & \frac{\varepsilon_N q_{N(N+1)}^{(N)}(0)}{\sqrt{|q_{NN}^{(N)}(0)|}} & \dots & \frac{\varepsilon_N q_{Nn}^{(N)}(0)}{\sqrt{|q_{NN}^{(N)}(0)|}} \\ & & & & 1 & & \\ & 0 & & & & \ddots & 0 \\ & & & & & & 1 \end{vmatrix} = \sqrt{|q_{NN}^{(N)}(0)|} \neq 0$$

Thus, the inverse function theorem [$\mathcal{H}$S, Thm. 2.10] gives the existence of open sets $V_{N+1} \subset W$, $\mathcal{U}_{N+1} \subset \mathbb{R}^n$ such that $0 \in V_{N+1}$, $0 = \mathcal{H}(0) \in \mathcal{U}_{N+1}$ and $\mathcal{H}|_{V_{N+1}}$ is a diffeomorphism $V_{N+1} \to \mathcal{U}_{N+1}$

Hence

$$\Big(\underbrace{\varphi\circ\mathcal{H}_N^{-1}\circ\mathcal{H}|_{V_{N+1}}^{-1}}_{\substack{\mathcal{H}_{N+1}^{-1}\\ \text{diffeomorphism}}}\Big)(y)=\sum_{m=1}^{N-1}\varepsilon_m y_m^2+\varepsilon_N y_N^2+\sum_{N+1\le i,j\le n}\underbrace{\left(q_{ij}^{(N)}-\frac{q_{Ni}^{(N)}q_{Nj}^{(N)}}{q_{NN}^{(N)}}\right)\circ\mathcal{H}|_{V_{N+1}}^{-1}(y)}_{q_{ij}^{(N+1)}(y)}\,y_iy_j$$

$$=\sum_{m=1}^{N}\varepsilon_m y_m^2+\sum_{N+1\le i,j\le n}q_{ij}^{(N+1)}(y)\,y_iy_j,$$

where $q_{ij}^{(N+1)}$ satisfies $i_{N+1})-iii_{N+1})$.

[i_{N+1}) Trivial

ii_{N+1}) Trivial

iii_{N+1}) By the chain rule

$$\big(\partial_\ell\partial_k(\varphi\circ\mathcal{H}_{N+1}^{-1})(0)\big)=[D\mathcal{H}_{N+1}^{-1}(0)]^T\big(\partial_i\partial_j\varphi(0)\big)\,D\mathcal{H}_{N+1}^{-1}(0)$$

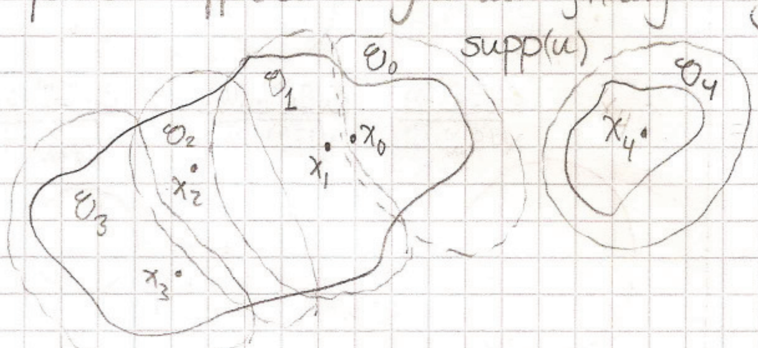
$$\|$$

$$\begin{pmatrix} 2\varepsilon_1 & & & \\ & \ddots & & O \\ & & 2\varepsilon_N & \\ & O & & \end{pmatrix}$$

so $[iii_{N+1})]^{\neg}$ would imply that $\det(\partial_i\partial_j\varphi(0))=0$]

Remark

The compact set $\operatorname{supp} u$ can only contain finitely many (non-degenerate) stationary points of φ.



Let $\{\mathcal{O}_j\}_{j=0}^N$ be a bounded open cover of $\operatorname{supp} u$ such that $\mathcal{O}_j$ contains one and only one stationary point of φ.

Partition of unity [GG, Thm. 2.17]: $\displaystyle\sum_{j=0}^N \psi_j=1$ on $\operatorname{supp}(u)$ with $\psi_j\in C_c^\infty(\mathcal{O}_j;[0,1])$

Stationary Phase

Then

$$I_{u,\varphi}(\lambda) = \sum_{j=0}^{N} \int_{\mathbb{R}^n} \overbrace{\psi_j}^{C_c^\infty(O_j) \ni} u \, e^{i\lambda\varphi} dm = \sum_{j=0}^{N} I_{u\psi_j,\varphi}(\lambda)$$

so we can assume that φ has one and only one non-degenerate stationary point in supp u.

The Morse Lemma inspires us to consider the case $\varphi(x) = \langle x, Ax \rangle$, where A is a real, symmetric and invertible $n\times n$-matrix.

Proposition

Let A be a real, symmetric and invertible $n\times n$-matrix. Then for all $u \in C_c^\infty(\mathbb{R}^n)$, $\lambda > 0$ and all integers $k > 0$ and $s > \frac{n}{2}$

$$\left| I_{u,\langle x,Ax\rangle}(\lambda) - \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \sum_{j=0}^{k-1} \frac{\langle D, A^{-1}D\rangle^j u(0)}{(4i)^j j!} \lambda^{-\frac{n}{2}-j} \right| \le C_k \left(\frac{\|A^{-1}\|}{\lambda}\right)^{\frac{n}{2}+k} \sum_{|\alpha| \le s+2k} \|D^\alpha u\|_{L^2}$$

where $D = \frac{1}{i}(\partial_1, \ldots, \partial_n)$.

Lemma

Let A be a real, symmetric and invertible matrix.

$$\mathcal{F}(e^{i\lambda\langle x,Ax\rangle})(\xi) = \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}}$$

Proof of proposition:

Note that

$$\begin{aligned}
I_{u,\langle x,Ax\rangle}(\lambda) &= \int_{\mathbb{R}^n} u(x) e^{i\lambda\langle x,Ax\rangle} dx \\
&= \langle e^{i\lambda\langle x,Ax\rangle}, \mathcal{F}(2\pi)^{-n} \bar{\mathcal{F}} u \rangle \\
&= \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}} \underbrace{\bar{\mathcal{F}}u(\xi)}_{\mathcal{F}u(-\xi)} d\xi \\
&= \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}} \mathcal{F}u\right)(0)
\end{aligned}$$

Stationary Phase

so

$$\left| I_{u,\langle x,Ax\rangle}(\lambda) - \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \sum_{j=0}^{k-1} \lambda^{-\frac{n}{2}-j} \frac{\langle D, A^{-1}D\rangle^j u(0)}{(4i)^j j!} \right|^2 .$$

$$= \underbrace{\left|\det\left(\frac{A}{\pi i}\right)\right|^{-1}}_{\propto |\det(A^{-1})| \leq \|A^{-1}\|^n} \lambda^{-n} \left| \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}} \mathcal{F}u\right)(0) - \sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle D, A^{-1}D\rangle^j u}{(4i)^j j!}(0)\right|^2$$

$$\lesssim \left(\frac{\|A^{-1}\|}{\lambda}\right)^n \left\| \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}} \mathcal{F}u\right) - \sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle D, A^{-1}D\rangle^j u}{(4i)^j j!} \right\|_\infty^2$$

Sobolev $\rightarrow$ $$\lesssim \left(\frac{\|A^{-1}\|}{\lambda}\right)^n \sum_{|\alpha|\leq s} \left\| D^\alpha \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}} \mathcal{F}u\right) - D^\alpha \sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle D, A^{-1}D\rangle^j}{(4i)^j j!} u \right\|_{L^2}^2$$

Parseval $\rightarrow$ $$\approx \left(\frac{\|A^{-1}\|}{\lambda}\right)^n \sum_{|\alpha|\leq s} \left\| \underbrace{\left| e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}} - \sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle \xi, A^{-1}\xi\rangle^j}{(4i)^j j!} \right|} \mathcal{F} D^\alpha u \right\|_{L^2}^2$$

with $w = -i\frac{\langle \xi, A^{-1}\xi\rangle}{4\lambda}$ $$\rightarrow = \left| e^w - \sum_{j=0}^{k-1} \frac{w^j}{j!} \right| \overset{\text{Taylor}}{\leq} \left| \frac{k}{k!} w^k \int_0^1 (1-\theta)^{k-1} e^{\theta w} dw \right| \overset{w \text{ imaginary}}{\leq} \frac{|w|^k}{k!}$$

$$\lesssim \left(\frac{\|A^{-1}\|}{\lambda}\right)^n \sum_{|\alpha|\leq s} \left\| \underbrace{\left|\frac{\langle \xi, A^{-1}\xi\rangle}{\lambda}\right|^k}_{\leq \left(\frac{\|A^{-1}\|}{\lambda}\right)^k |\xi|^{2k}} \mathcal{F} D^\alpha u \right\|_{L^2}^2$$

$$\lesssim \left(\frac{\|A^{-1}\|}{\lambda}\right)^{n+2k} \sum_{|\alpha|\leq s+2k} \|D^\alpha u\|_{L^2}^2$$

Thus, the desired result follows by taking squareroots and using that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$. ▨

Proof of lemma:

From exercise 1.1 we know

If B is symmetric, unitarily diagonalizable and invertible with $\operatorname{Re} B \geq 0$ then

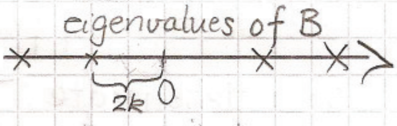
$$\mathcal{F}\left(e^{-\langle x, Bx\rangle}\right)(\xi) = \frac{\pi^{n/2}}{(\det(B))^{1/2}} e^{-\frac{1}{4}\langle \xi, B^{-1}\xi\rangle}.$$

After showing ✓ the lemma follows by setting $B = -i\lambda A$.

Stationary Phase

Let B be symmetric, unitarily diagonalizable and invertible with $\operatorname{Re} B \geq 0$. If $\mu_1, \dots, \mu_n$ denotes the eigenvalues of B we set

$$k = \frac{\min\{|\mu_1|, \dots, |\mu_n|\}}{2}$$



Then for $0<\varepsilon<k$ the matrix $B+\varepsilon I$ is symmetric, unitarily diagonalizable and inv w/ $\operatorname{Re}(B+\varepsilon I)>0$, whereby

$$(**) \qquad \mathcal{F}\left(e^{-\langle x, (B+\varepsilon I)x\rangle}\right)(\xi) = \frac{\pi^{n/2}}{(\det(B+\varepsilon I))^{1/2}} e^{-\frac{1}{4}\langle \xi, (B+\varepsilon I)^{-1}\xi\rangle}$$

$$B = \underset{\text{unitary}}{\mathcal{U}} \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \mathcal{U}^{-1} \longrightarrow \;\; \|$$

$$\frac{\pi^{n/2}}{\left(\prod_{j=1}^{n}(\mu_j+\varepsilon)\right)^{1/2}} e^{-\frac{1}{4}\sum_{j=1}^{n}(\mu_j+\varepsilon)^{-1}[\mathcal{U}^{-1}\xi]_j^2}$$

Note that we have the pointwise limits

$$(***) \qquad e^{-\langle x, (B+\varepsilon I)x\rangle} \xrightarrow[\varepsilon\to 0^+]{} e^{-\langle x, Bx\rangle}$$

$$\frac{\pi^{n/2}}{(\det(B+\varepsilon I))^{1/2}} e^{-\frac{1}{4}\langle \xi, (B+\varepsilon I)^{-1}\xi\rangle} \xrightarrow[\varepsilon\to 0^+]{} \frac{\pi^{n/2}}{(\det B)^{1/2}} e^{-\frac{1}{4}\langle \xi, B^{-1}\xi\rangle}$$

Moreover,

$$\left|e^{-\langle x, (B+\varepsilon I)x\rangle}\right| = e^{-\operatorname{Re}\langle x, Bx\rangle - \varepsilon\|x\|^2} \leq 1,$$

$$\left|\frac{\pi^{n/2}}{(\det(B+\varepsilon I))^{1/2}} e^{-\frac{1}{4}\langle \xi, (B+\varepsilon I)^{-1}\xi\rangle}\right| \leq \left(\frac{\pi}{k}\right)^{n/2} e^{-\frac{1}{4}\langle \xi, (B+kI)^{-1}\xi\rangle} \in C^\infty$$

where we use that $\left|\left(\prod_{j=1}^{n}(\mu_j+\varepsilon)\right)^{1/2}\right| = (\underbrace{|\mu_1+\varepsilon|}_{\geq k} \cdots \underbrace{|\mu_n+\varepsilon|}_{\geq k})^{1/2} \geq k^{n/2}$ and that $(\mu_j+k)^{-1} \leq (\mu_j+\varepsilon)^{-1}$ for $j \in \{1, \dots, n\}$.

By dominated convergence $(***)$ therefore holds in $\mathcal{S}'$ and so the LHS of $(**)$ goes to $\mathcal{F}(e^{-\langle x, Bx\rangle})$ in $\mathcal{S}'$ (and thereby also in $\mathcal{D}'$) as $\varepsilon \to 0^+$. Similarly, the RHS of $(**)$ goes to $\frac{\pi^{n/2}}{(\det B)^{1/2}} e^{-\frac{1}{4}\langle \xi, B^{-1}\xi\rangle}$ in $\mathcal{D}'$ as $\varepsilon \to 0^+$. The desired result follows. $\blacksquare$

Stationary Phase

Principle of stationary phase

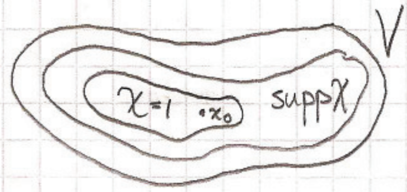
Let $u \in C_c^\infty(\mathbb{R}^n)$ and consider a $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ with one and only one stationary point x_0 in $\operatorname{supp} u$; this is assumed to be non-degenerate. Then for all integers $k > 0$ we have

$$\left| I_{u,\varphi}(\lambda) - e^{i\lambda\varphi(x_0)} \sum_{j=0}^{k-1} T_j u(0) \lambda^{-\frac{n}{2}-j} \right| \leq C_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-k}$$

where T_j is a differential operator of order $2j$ with C^∞-coefficients.

Proof:

Let $\mathcal{H}: V \to U$ be as in the Morse Lemma and choose $\chi \in C_c^\infty(V)$ with $\chi = 1$ near x_0



Then

$$I_{u,\varphi}(\lambda) = \int_V e^{i\lambda\varphi(x)} (\chi u)(x)\,dx + \int_{\mathbb{R}^n} e^{i\lambda\varphi(x)} [(1-\chi)u](x)\,dx$$

$$= \int_U e^{i\lambda \overbrace{\varphi \circ \mathcal{H}^{-1}(x)}^{=\varphi(x_0) + \langle x, \Xi x\rangle}} \underbrace{(\chi u)\circ\mathcal{H}^{-1}(x) \left|\det J\mathcal{H}^{-1}(x)\right|}_{= f_u(x) \in C_c^\infty(\mathbb{R}^n)} dx + I_{(1-\chi)u,\varphi}(\lambda)$$

$$= e^{i\lambda\varphi(x_0)} I_{f_u, \langle x, \Xi x\rangle}(\lambda) + I_{(1-\chi)u,\varphi}(\lambda)$$

so by setting

$$T_j u = \left(\det\left(\frac{\Xi}{\pi i}\right)\right)^{-\frac{1}{2}} \frac{\langle D, \Xi^{-1} D\rangle^j f_u}{(4i)^j j!}$$

and letting s be the smallest integer $> \frac{n}{2}$ we get

$$\left| I_{u,\varphi}(\lambda) - e^{i\lambda\varphi(x_0)} \sum_{j=0}^{k-1} T_j u(0) \lambda^{-\frac{n}{2}-j} \right|$$

$$\leq \left| I_{f_u, \langle x, \Xi x\rangle}(\lambda) - \left(\det\left(\frac{\Xi}{\pi i}\right)\right)^{-\frac{1}{2}} \sum_{j=0}^{k-1} \frac{\langle D, \Xi^{-1} D\rangle^j f_u(0)}{(4i)^j j!} \lambda^{-\frac{n}{2}-j} \right| + \left| I_{(1-\chi)u,\varphi}(\lambda) \right|$$

$$\leq C_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-k}$$

$\blacksquare$

Stationary Phase

Remark:

Observe that by definition of T_j and f_u we have

$$T_0 u(0) = \left(\det\left(\frac{\Sigma}{\pi i}\right)\right)^{-\frac{1}{2}} f_u(0) = \underbrace{\left(\det\left(\frac{\Sigma}{\pi i}\right)\right)^{-\frac{1}{2}} \left|\det J\mathcal{H}^{-1}(0)\right|}_{= C_{(\partial_i \partial_j \varphi(x_0))_{ij}}} u(x_0)$$

so

$$\left| I_{u,\varphi}(\lambda) - C_{(\partial_i\partial_j\varphi(x_0))_{ij}}\, e^{i\lambda\varphi(x_0)} u(x_0) \lambda^{-\frac{n}{2}} \right| \leq C_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-1}.$$