1

TS 13/06

# Thomas - Stein restriction Theorem

The restriction problem asks whether it is possible to restrict the Fourier transform of a function to the unit sphere (or any other hypersurface) $S \subseteq \mathbb{R}^n$. I.e, it asks for estimates

$$\boxed{\|\hat{f}\|_{L^q(S)} \leq C\|f\|_p \text{ with the surface measure } d\sigma \text{ on } S.} \quad *$$

And the problem is to decide for which $p, q$ such estimates exist.

The problem is for a large part open, but the 'Restriction conjecture for the sphere' asserts that $*$ holds if and only if

$$\boxed{p' \geq \frac{n+1}{n-1} q} \ \text{(I)} \quad \& \quad \boxed{p < \frac{2n}{n+1}} \ \text{(II)}$$

Both (I) & (II) are known to be neccessary. We will see this for (II) and for (I) it is similar to the very first exercise in the course (1.1(5))

The Thomas-Stein restriction Theorem confirms the conjecture in the case $q=2$, where it works out to

TS 13/06

**Theorem (Thomas-Stein)** If $1 \le p \le 2(n+1)/_{n+3}$ then
$$\|\hat{f}\|_{L^2(S^{n-1})} \le C \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n)$$

To prove it, we employ a '$TT^*$-method' approach - to that end we need a little extra theory about the Fourier transform.

## Fourier transforms & measures

**Definition** Let $\mu$ be a measure on $\mathbb{R}^n$ with finite total variation and let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be Schwartz. Define

- Fourier transform of $\mu$
$$\hat{\mu} : \mathbb{R}^n \to \mathbb{C} \; : \; \hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \, d\mu(x)$$
- Convolution of $\mu$ & $\varphi$
$$\varphi * \mu : \mathbb{R}^n \to \mathbb{C} \; : \; \varphi * \mu(x) = \int_{\mathbb{R}^n} \varphi(x-y) \, d\mu(y)$$

(Both are well-defined & bounded since $|\mu|(\mathbb{R}^n) < \infty$)

Now we get (using Fubini-Tonelli many times) the following lemmas:

Lemma $\mu, \nu$ finite variation measures

$\varphi$ Schwartz :

(1) $\widehat{\varphi \check{\mu}} = \hat{\varphi} * \mu$

(2) $\widehat{\varphi \mu} = \hat{\varphi} * \hat{\mu}$

(3) $\int \hat{\mu}\, d\nu = \int \hat{\nu}\, d\mu$

Corollary $\mu$ finite variation measure, $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$\int \hat{f}\, \overline{\hat{g}}\, d\mu = \int (\hat{\mu} * \bar{g}) \cdot f\, dx \quad \text{i.e.}$$

$$\langle \hat{f}, \hat{g} \rangle_{L^2(\mu)} = \langle f, g * \overline{\hat{\mu}} \rangle_{L^2(\mathbb{R}^n)}$$

---

Now we are ready for the first simplification – the $TT^*$-method

We need to prove $\|\hat{f}\|_{L^2(S^{n-1})} \leq C \|f\|_{L^p(S^{n-1})}$ say $\forall f \in \mathcal{S}(\mathbb{R}^n)$, i.e.

$$\int |\hat{f}(\xi)| \, d\sigma(\xi) \leq C^2 \|f\|_p^2$$

$$\int |\hat{f}(\xi)|^2 d\sigma(\xi) = \langle \hat{f}, \hat{f} \rangle_{L^2(\sigma)} \overset{\text{cor.}}{=} \langle f, \overline{\hat{\sigma}} * f \rangle = \int f \cdot \hat{\sigma} * \bar{f}\, dx$$

By Hölders inequality, it is enough to prove

$\|\hat{\sigma} * f\|_{L^{p'}} \leq \tilde{C} \|f\|_{L^p}$. To obtain such estimates we need to investigate the decay of $\hat{\sigma}$.

For this we use the method of stationary phase

Recall

TS 13/06

Principle of non-stationary phase

$u \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ w. $\nabla\varphi \neq 0$ on supp $u$.

Then $\forall N \in \mathbb{N}$ $\exists C_{N,u,\varphi}$ s.t.

$$\left| \int_{\mathbb{R}^n} u(x) \exp(i\lambda\varphi(x))\, dx \right| \leq C_{N,u,\varphi} \lambda^{-N}, \quad \lambda > 0$$

Further:

Principle of Stationary phase

$u \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with only one stationary point $x_0 \in$ supp $u$ which is non-degenerate. Then

$$\int_{\mathbb{R}^n} u(x) e^{i\lambda\varphi(x)}\, dx = C_\varphi u(x_0) e^{i\lambda\varphi(x_0)} \lambda^{-n/2} + \mathcal{O}(\lambda^{-(n+1)/2})$$

as $\lambda \to \infty$.

---

Proposition let $\sigma$ be the surface measure of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. then for $|x| \gg 1$ we have

$$\hat{\sigma}(x) = C \frac{e^{2\pi i|x|}}{|x|^{(n-1)/2}} + C \frac{e^{-2\pi i|x|}}{|x|^{(n-1)/2}} + \mathcal{O}(|x|^{-n/2})$$

Proof by rotational symmetry of $\sigma$, and hence of $\hat{\sigma}$, it is enough to estimate

$$\hat{\sigma}(\lambda e_n) = \int_{S^{n-1}} e^{-2\pi i\lambda\omega_n}\, d\sigma(\omega)$$

TS 13/06

The function $w_n : S^{n-1} \to [-1,1]$ is stationary on the north & south poles only, and these are seen to be non-degenerate. If we let $\psi_+, \psi_-$ be $C^\infty$-cutoff-functions supported near the north/south poles respectively, we get

$$\hat{\sigma}(\lambda e_n) = \int_{S^{n-1}} \psi_+(w) e^{-2\pi i \lambda w_n} d\sigma(w) + \int_{S^{n-1}} \psi_-(w) e^{-2\pi i \lambda w_n} d\sigma(w) +$$
$$+ \int_{S^{n-1}} (1 - \psi_+(w) - \psi_-(w)) e^{-2\pi i \lambda w_n} d\sigma(w)$$

The contribution for the first two terms are

$$C \psi_\pm(w) e^{-2\pi i \lambda w_n} \lambda^{-(n-1)/2} \Big|_{w = \pm e_n} + O(\lambda^{-n/2})$$

(the integral is in $n-1$ dimensions) by the principle of stationary phase, and $O(\lambda^{-n/2})$ by the principle of non-stationary phase. Thus

$$\hat{\sigma}(\lambda e_n) = C \frac{e^{2\pi i \lambda}}{2\pi} \lambda^{-(n-1)/2} + C e^{-2\pi i \lambda} \lambda^{-(n-1)/2} + O(\lambda^{-n/2})$$

$\square$

In particular

$|\hat{\sigma}(x)| \lesssim |x|^{-(n-1)/2}$. We are now ready to attack the theorem in the region $p < \frac{2n+2}{n+3}$ (the case $p = \frac{2n+2}{n+3}$ is more difficult, and will be dealt with later, if time allows)

Proof ($p < \frac{2n+2}{n+3}$) TS 13/06

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be $1$ near $0$ and define

$\psi_k(x) = \varphi(2^{-k}x) - \varphi(2^{-(k-1)}x)$ (i.e. $\psi_k$ is supported on an 'annulus' $|x| \sim 2^k$). We then have

$1 = \varphi(x) + \sum_{k>0} \psi_k(x)$ (for each $x$ only finitely many terms are non-zero)

Hence we get

$f * \hat{\sigma} = f * (\varphi \hat{\sigma}) + \sum_{k>0} f * (\psi_k \hat{\sigma})$, hence ($\Delta$-inequality)

$\|f * \hat{\sigma}\|_{p'} \le \|f * (\varphi \hat{\sigma})\|_{p'} + \sum_{k>0} \|f * (\psi_k \hat{\sigma})\|_{p'}$

The first term can be easily handled by e.g. Young's inequality ($\|f * (\psi_k \hat{\sigma})\|_{p'} \le \|f\|_p \|\psi_k \hat{\sigma}\|_{p/2(p-1)}$)

We need something like

$\|f * (\psi_k \hat{\sigma})\|_{p'} \le 2^{-\varepsilon k} \|f\|_p$ $(*)$ for some $\varepsilon > 0$: To get this we use interpolation $\boxed{L^1 \to L^\infty, L^2 \to L^2}$

Since $\|\psi_k \hat{\sigma}\|_\infty \lesssim 2^{-(n-1)k/2}$ by the preceding and the fact that $\psi_k$ is supported on $|x| \sim 2^k$ we get (From a trivial case of Young's inequality $r = \infty$, $p = 1$, $q = \infty$)

$\boxed{\|\psi_k \hat{\sigma}\|_\infty \lesssim 2^{-(n-1)k/2}}$ (constants not depending on $k$)

For the $L^2 \to L^2$ bound note that

$\|f * g\|_2 \le \|\hat{g}\|_\infty \|f\|_2$ just from Plancherel, Hölder and the fact that $\widehat{f * g} = \hat{f}\hat{g}$

Hence we need to prove an estimate for  TS - 13/06

$\|\widehat{\psi_k \cdot \hat\sigma}\|_\infty = \|\hat\psi_k * \sigma\|_\infty$. By the scaling properties of the ~~Fourier transform~~ $\hookrightarrow$ we need $\|\hat\psi_k * \sigma\|_\infty \lesssim 2^k$

$\boxed{\hat\psi_k(x) = 2^{nk}\hat\psi_0(2^k x).}$ Since $\psi_0$ is Schwartz, $\hat\psi_0$ is ~~Schwartz, and we get~~

$|\hat\psi_k(x)| \lesssim \dfrac{2^{nk}}{(1+2^k|x|)^N}$ (constants indep of $k$). By $\Delta$-ineq.

It suffices to show

$\dfrac{2^{nk}}{(1+|x|2^k)^N} * d\sigma(x) \lesssim 2^k$ – this can be done

dividing $\mathbb{R}^n$ into annuli around $x$ with radius $\sim 2^{-k+j}$ and summing in $j$.

Interpolating the $1 \to \infty$ & $2 \to 2$ bound with $\Theta = \frac{2}{p'}$ gives $(*)$ and we are done. $\square$

The end point estimate $p = \frac{2n+2}{n+3}$ is more difficult: we need the more sophisticated interpolation theorem "Stein interpolation thm" where the operator is allowed to depend analytically on the interpolation parameter

$$\left[\; T_z f = \sum_{k>0} 2^{-\frac{(n+1)}{2}\left(z - \frac{n-1}{n+1}\right)k} f * (\psi_k \hat\sigma) \;\right]$$

By the Stein interpolation theorem it suffices to prove

TS, 13/06

$$\Big\|\sum_{k>0} 2^{[\frac{n-1}{2}+it]k} f*(\psi_k\hat{\sigma})\Big\|_\infty \lesssim \|f\|_1 \qquad (\ast) \quad \text{and}$$

$$\Big\|\sum_{k>0} 2^{[-1+it]k} f*(\psi_k\hat{\sigma})\Big\|_2 \lesssim \|f\|_2 \qquad (\ast\ast)$$

$(\ast)$ is easy, since it suffices to show that

$\Big\|\sum_{k>0} 2^{[\frac{n-1}{2}+it]k}\psi_k\hat{\sigma}\Big\|_\infty \lesssim 1$ which follows from the fact that $\sum_{k>0} 2^{[\frac{n-1}{2}+it]}\psi_k(x) = \mathcal{O}(|x|^{n-1/2})$ by construction of the $\psi_k$'s and the decay estimate of $\hat{\sigma}$.

For $(\ast\ast)$ as in the previous proof we need to prove

$\Big\|\sum_{k>0} 2^{[-1+it]k}\hat{\psi}_k*\sigma\Big\|_\infty \lesssim 1$ – here we finally give in to the Δ-triangle inequality and prove that

$\sum 2^{-k}\|\hat{\psi}_k*\sigma\|_\infty \lesssim 1.$

Note how the estimate $\|\psi_k*\sigma\|_\infty \lesssim 2^k$ does not suffice here. We prove

$$|\hat{\psi}_k*\sigma(x)| \lesssim \begin{cases} \dfrac{2^k}{(2^k d(x,S))^N} & d(x,S)\geq 2^{-k} \\ 1+2^{2k}d(x,S) & d(x,S)\leq 2^{-k} \end{cases}$$

The first one can be seen in the same way as the estimate in the last proof, decomposing the sphere into "annular" regions $d(x,\omega) \sim 2^j d(x,S)$, using the schwartz-estimate

| TS 13/06

$$\hat{\varphi}_k(x) \lesssim \frac{2^{nk}}{(1+2^k|x|)^N} \quad (*)$$

and summing in $j$. The difficult case is $d(x,S) \le 2^{-k}$, where $(*)$ is not very good.

We first observe that
$|\nabla(\hat{\varphi}_k * \sigma)(x)| \lesssim 2^{2k}$ since
$\nabla(\hat{\varphi}_k * \sigma)(x) = 2^k (2^{-k}\nabla\hat{\varphi}) * \sigma(x)$ and
$2^{-k}\nabla\hat{\varphi}_k$ satisfies $(*)$ as well (with another implied constant).

From the fundamental theorem of calculus, it suffices to prove (⊛⊛) on the unit sphere.
By rotational symmetry take $x = e_1$ – that is, we need

$$\left| \int_{S^{n-1}} \hat{\varphi}_k(e_1 - \omega)\, d\sigma(\omega) \right| = O(1).$$

For $|e_1 - \omega|$ bounded away from $0$ we can use $(*)$ again, so we only need a region $|e_1 - \omega| \ll 1$
– here we can parametize $\omega$ as
$\omega = (\underline{\omega}, (1-|\underline{\omega}|^2)^{1/2})$, $|\underline{\omega}| \ll 1$. The integral is now

$$\int_{|w|\ll 1} \hat{\psi}_k\left(\underline{w}, 1-(1-|w|^2)^{1/2}\right) J(w)\, d\underline{w} = \int_{|w|\ll 1} \hat{\psi}_k\left(\underline{w}, O(|w|^2)\right)\left(1+O(|w|^2)\right) d\underline{w}$$

By $(*)$ the contribution of $\int_{|w|\geq 1} \hat{\psi}_k(\underline{w}, O(|w|^2))(1+O(|w|^2))\,dw$ is $O(1)$.

Hence we may estimate

$$\int_{\mathbb{R}^{n-1}} \hat{\psi}(\underline{w}, O(|w|^2))(1+O(|w|^2))\,dw$$ – we now prove that this is

$$C\int_{\mathbb{R}^{n-1}} \hat{\psi}_k(w,0)\,dw + O(1)$$ – If this is the case we can just choose the cutoff-function such that $\int_{\mathbb{R}^{n-1}} \hat{\psi}_0(\underline{w},0)\,d\underline{w} = \int_{\mathbb{R}^{n-1}} \hat{\psi}_k(\underline{w},0)\,d\underline{w} = 0$ and we would be done.

To prove it, use Taylors formula to get

$$\hat{\psi}_K(\underline{w}, O(|\underline{w}|^2)) = \hat{\psi}_K(\underline{w},0) + O\left(\frac{2^{(n+1)k}|\underline{w}|^2}{(1+2^k|w|)^N}\right)$$

(since $2^{-k}\nabla\hat{\psi}_k$ satisfies $(*)$) : The discrepancy between our two terms is

$$\int O\left(\frac{2^{(n+1)k}|\underline{w}|^2}{(1+2^k|w|)^N}\right) d\underline{w}$$ which can be calculated directly to be $O(1)$ (choose $N=n+2$ e.g.)

This completes the proof.