Exercises 7 – Useful facts

(1) a) Let H be a Hilbert space and $[\cdot,\cdot] : H\times H \to \mathbb{C}$ sesquilinear and such that $[h,h]\geq 0 \ \forall h\in H$. Show the Cauchy-Schwarz inequality $|[f,g]| \leq [f,f]^{1/2}[g,g]^{1/2} \ \forall f,g\in H$.

b) Let M be a subset of a Banach space X and $\operatorname{span} M := \{\sum a_i m_i : a_i\in\mathbb{C},\ m_i\in M\}$. Show that

$\operatorname{span} M$ is dense in X $\iff$ Every continuous linear functional on X, which vanishes on M is 0

c) Let A be a Banach algebra, $\varphi: A\to\mathbb{C}$ linear and multiplicative ($\varphi(ab)=\varphi(a)\varphi(b)$). Show that φ is continuous and $\|\varphi\|\leq 1$. If A is unital $\Rightarrow \|\varphi\|\in\{0,1\}$.

Hint: If $\exists\, a\in A$ s.t. $\|a\|<1$, but $\varphi(a)=1 \Rightarrow \varphi\left(\sum_{n=1}^{\infty} a^n\right) = 1+\varphi\left(\sum_{n=1}^{\infty} a^n\right)$

d) Let G be a locally compact abelian group (say $G=\mathbb{R}$) and $h\in L^1(G)$. Show that $\|\underbrace{h*\cdots*h}_{n \text{ factors}}\|_{L^1(G)}^{1/n} \xrightarrow{n\to\infty} \|\hat{h}\|_{L^\infty(\hat{G})}$.

Hint: In a (complete) Banach algebra $\lim_{n\to\infty}\|a^n\|^{1/n} = \sup|\sigma(a)|$.

(2) a) Prove that $\mathbb{R}$ is a locally compact abelian group.

b) Prove that if $G_1,\dots,G_N$ are locally compact abelian groups, so is $G_1\times\dots\times G_N$ (with the product topology and component-wise multiplication).

c) Prove that an infinite-dimensional Banach space is not locally compact.

Remark: A vector space with a Hausdorff topology, such that multiplication with scalars and vector addition are continuous, is locally compact $\iff$ it is homeomorphic to $\mathbb{R}^n$ for some n.

(3) Let G be a locally compact abelian group and m the Haar measure on G. Show:

a) If $\emptyset \neq V \subset G$ open $\Rightarrow m(V) > 0$.

b) $E \subset G$ measurable $\Rightarrow m(E) = m(-E)$.

(4) a) Let $\gamma : \mathbb{R} \to \mathbb{C}$ continuous, $\gamma(x+y) = \gamma(x)\gamma(y)$ $\forall x, y \in \mathbb{R}$ and γ not identically 0. By continuity $\exists \delta > 0 : \int_0^\delta \gamma(x)\,dx = \alpha \neq 0$. Show $\alpha\,\gamma(x) = \int_x^{x+\delta} \gamma(t)\,dt$. Conclude that γ is differentiable and $\gamma'(x) = \gamma'(0)\,\gamma(x)$. Show that the only bounded solutions are $\gamma_y(x) := e^{iyx}$ for some $y \in \mathbb{R}$.

b) Endow the space of all such γ with the compact-open topology (as described in Rudin, 1.2.6). Show that $y \leftrightarrow \gamma_y$ is a homeomorphism.

c) Show that all bounded γ as in a), with $\mathbb{R}$ replaced by $\mathbb{R}/\mathbb{Z}$ are of the form $\gamma_n(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$.

(5) Let G be a locally compact abelian group. $\phi : G \to \mathbb{C}$ is said to be positive definite, if $\sum_{n,m=1}^{N} c_n \bar{c}_m \,\phi(x_n - x_m) \geq 0$ for all $x_1, \ldots, x_N \in G$, $c_1, \ldots, c_N \in \mathbb{C}$.

Let $f \in L^2(G)$, $\tilde{f}(x) = \overline{f(-x)}$. Show that $f * \tilde{f}$ is positive definite and $\in C_0(G)$.

Hint: Use Young's inequality to deduce the last assertion.