Exercises 6 – Calculus of pseudo differential operators

Recall that $S^m = \{ a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) : \exists C_{\alpha\beta} : |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$

and set $S^{-\infty} := \bigcap_{m\in\mathbb{R}} S^m$, $S^\infty := \bigcup_{m\in\mathbb{R}} S^m$.

① a) (<u>Borel's Lemma</u>) Let $(a_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{C}$; $(\varepsilon_j)_{j\in\mathbb{N}_0} \subseteq (0,\infty)$, $\varepsilon_j \downarrow 0$ sufficiently fast and $\eta \in C_c^\infty(\mathbb{R})$, $\eta(x)=1$ for $|x|\leq 1$ and $\eta(x)=0$ for $|x|\geq 2$.
Show that $f(x) = \sum_{j=0}^\infty \frac{a_j}{j!} x^j \eta(x/\varepsilon_j) \in C^\infty(\mathbb{R})$ and $\partial_x^j f(0) = a_j \ \forall j$.

b) (<u>asymptotic summation</u>) Let $a_j \in S^{m_j}$, $m_j \downarrow -\infty$, ε_j as above and $\varphi \in C^\infty(\mathbb{R}^d)$, $\varphi(x)=1$ for $|x|\geq 2$, $\varphi(x)=0$ for $|x|\leq 1$.
Show that $a(x,\xi) := \sum_{j=0}^\infty a_j(x,\xi)\,\varphi(\varepsilon_j \xi) \in S^{m_0}$ and

$(*)$ $a - \sum_{j=0}^k a_j \in S^{m_{k+1}}$ $\forall k \in \mathbb{N}_0$

<u>Notation:</u> We write $a \sim \sum_{j=0}^\infty a_j$ if $(*)$ holds. Observe that $a \sim 0$ if and only if $a \in S^{-\infty}$. Using 1b, we define an associative product $\# : S^\infty/S^{-\infty} \times S^\infty/S^{-\infty} \to S^\infty/S^{-\infty}$ via

$$a\#b \sim \sum_{|\alpha|\geq 0} \frac{i^\alpha}{\alpha!} D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi).$$

② a) Let $a \in S^m$, $b \in S^{\tilde m}$, $m, \tilde m \in \mathbb{R}\cup\{-\infty\}$. Show that $a+b \in S^{\max\{m,\tilde m\}}$, $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}$, $ab \in S^{m+\tilde m}$, $a\#b \in S^{m+\tilde m}/S^{-\infty}$.
If $|a(x,\xi)| \geq C\langle\xi\rangle^m$, show that $a^{-1} \in S^{-m}$ and $r := a\#a^{-1} - 1$, $\tilde r := a^{-1}\#a - 1 \in S^{-1}/S^{-\infty}$.

b) (<u>Neumann series</u>) Let $b \in S^{-1}$, $b^{\#n} = \underbrace{b\#\cdots\#b}_{n \text{ factors}}$.
Show that $(1 + b + b^{\#2} + b^{\#3} + \dots)\#(1-b) \sim 1$.

c) Continuing a), let $s \sim \sum_{j=1}^{\infty} r^{\#j}$, $\tilde{s} \sim \sum_{j=1}^{\infty} \tilde{r}^{\#j}$, $b = a^{-1} \# (1+s)$, $\tilde{b} = (1+\tilde{s}) \# a^{-1}$. Show that $\tilde{b} \# a \sim a \# b \sim 1$ and therefore $\tilde{b} \sim \tilde{b} \# a \# b \sim b$, or $a \# b - 1 \sim b \# a - 1 \sim 0$.

Interpretation: Given $a \in S^m$ elliptic ($|a| \geq C \langle \xi \rangle^m$), we have found an inverse $b \in S^{-m}$ of a up to a negligible remainder $\in S^{-\infty}$.

3 a) Given $a \in S^m$, define the adjoint of $a(x,D)$ with respect to the L^2-scalar product: $\forall f, g \in \mathcal{S}(\mathbb{R}^d)$: $\langle a(x,D)^* f, g \rangle_{L^2} := \langle f, a(x,D) g \rangle_{L^2}$.
Integrate by parts as in Ex. 5.2 to show that $a(x,D) : \mathcal{S} \to \mathcal{S}$ and conclude that $a(x,D)$ extends by duality, $\forall f \in \mathcal{S}\ \forall g \in \mathcal{S}'$ $\langle f, a(x,D) g \rangle := \langle a(x,D)^* f, g \rangle$, to an operator on $\mathcal{S}'$.

b) Let $a \in S^{-\infty}$. Show that $a, a^* : \mathcal{E}' \to \mathcal{S}$ and therefore, by duality, also $a^*, a : \mathcal{S}' \to C^\infty$.

Hint: Integrate by parts yet again.