Exercises 5 — Integral operators from PDE

① a) Let $\psi : [0,\infty) \to [0,\infty]$ nonincreasing and $\Psi : \mathbb{R}^n \to [0,\infty]$ defined by $\Psi(x) := \psi(|x|)$. Show $\forall f \in L^p(\mathbb{R}^n)$:

$$|f * \Psi(x)| \leq \left(\int_{\mathbb{R}^n} \Psi\right) \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

and $|f * \Psi_t(x)| \leq \left(\int_{\mathbb{R}^n} \Psi\right) \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \quad \forall t>0.$

Here, $\Psi_t(x) = t^{-n}\, \Psi(x/t)$.

<u>Hint</u>: First take $\psi = \sum_{j=1}^{N} a_j \mathbb{1}_{(-r_j, r_j)}$, $a_j, r_j \in (0,\infty)$.
An arbitrary ψ can be approximated by such sums.

b) Let $\phi \in L^1(\mathbb{R}^n)$, $\int \phi = 1$, $|\phi(x)| \leq \psi(|x|) \in L^1$ with $\psi : [0,\infty) \to [0,\infty)$ as in a) and bounded. Show

$$\sup_{t>0} |f * \phi_t(x)| \leq C_\phi \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

c) Check that the proof of Lebesgue's differentiation thm. yields $\lim_{t\to 0^+} f * \phi_t(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

d) Let $\phi_1(x) = c_1 (1+|x|^2)^{-\frac{n+1}{2}}$, $\phi_2(x) = c_2 e^{-\frac{|x|^2}{4}}$, c_1, c_2 s.t. $\int \phi_{1/2} = 1$. Given $f \in L^p(\mathbb{R}^n)$, let $u_1(t,x) := f * (\phi_1)_t(x)$ and $u_2(t,x) := f * (\phi_2)_{\sqrt{t}}(x)$. Then

$\Delta u_1(t,x) = 0$ and $(\partial_t - \Delta)\, u_2(t,x) = 0$ for $t>0, x \in \mathbb{R}^n$;

and $\lim_{t\to 0^+} u_{1/2}(t,x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

(2) Let $S^m := \{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \forall \alpha, \beta \in \mathbb{N}_0^n \; \exists C_{\alpha\beta} : |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-|\alpha|} \}$

The operator $L_\xi := (1+|x|^2)^{-1}(1-\Delta_\xi)$ satisfies $L_\xi^N e^{ix\xi} = e^{ix\xi} \; \forall N$.

a) Check that for $a \in S^m$,

$$\mathrm{op}(a) f(x) := \int d\xi \; a(x,\xi) \hat{f}(\xi) e^{ix\xi} \overset{!}{=} \int d\xi \; L_\xi^N [a(x,\xi) \hat{f}(\xi)] e^{ix\xi}$$

defines an operator $\mathrm{op}(a) : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$. If a is a polynomial in ξ, $\mathrm{op}(a)$ is a differential operator.

b) Write $\mathrm{op}(a) f(x) = (k(x,\cdot) * f)(x)$, i.e.

$k(x,\cdot) = \mathcal{F}_\xi^{-1} a(x,\xi)$, a distribution $\in S'(\mathbb{R}^n) \; \forall x \in \mathbb{R}^n$.

Show that $k(x,\cdot)$ is a function away from the origin $z=0$ and satisfies $|k(x,z)| \leq C_N |z|^{-N}$ for $|z| \geq 1$, $N \in \mathbb{N}$, uniformly in x.

<u>Hint:</u> $\partial_\xi^\alpha a(x,\xi)$ is integrable in ξ for $|\alpha| \geq m+n+1$

and $|\mathcal{F}_{\xi \to z}^{-1} \partial_\xi^\alpha a(x,\xi)| = |z^\alpha k(x,z)|$.

c) Using the notation from b) and $n=1$, what is $k(x,\cdot)$ for $a(x,\xi) = \sum_{j=0}^{m} a_j \xi^j$, $a_j \in \mathbb{C}$?

<u>Hint:</u> What is $\delta^{(j)} * f$?

<u>Remark:</u> Operators of the form $\mathrm{op}(a)$, $a \in S^m$, are called "pseudo differential operators". In Dif Fun 1, we only considered x-independent symbols.