Exercises 4 — Miscellanea on Chapter 3

(1) TT^*-trick: Let H be a Hilbert and X a normed vector space, $T : H \to X$ linear and continuous and T^* the adjoint of T. Show:

$$\|T\|_{H\to X} = \|T^*\|_{X^*\to H} = \|TT^*\|^{1/2}_{X^*\to X}.$$

(2) a) Let $f \in L^1_{loc}(\mathbb{R}^d)$. $x \in \mathbb{R}^d$ is called a Lebesgue point of f provided that $\exists c \in \mathbb{C}$: $\lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f-c| = 0$. Show that a.e. $x \in \mathbb{R}^d$ is a Lebesgue point and that $c = f(x)$ for a.e. x.

b) Fundamental theorem of calculus: Let $f \in L^1_{loc}(\mathbb{R})$, $F(x) := \int_0^x f(y)\,dy$. Show that F is differentiable at every Lebesgue point of f and that $F' = f$ a.e.

(3) Let $(X, \tilde{\mathcal{B}}, \mu)$ a measure space and $\mathcal{B}$ a σ-finite σ-subalgebra of $\tilde{\mathcal{B}}$. The orthogonal projection from $L^2(X,\tilde{\mathcal{B}},\mu)$ to its closed subspace $L^2(X,\mathcal{B},\mu)$ is denoted by $E(\cdot,\mathcal{B})$.

Show that

a) $$\int_X f\,\overline{E(g,\mathcal{B})}\,d\mu = \int_X E(f,\mathcal{B})\,\bar{g}\,d\mu = \int_X E(f,\mathcal{B})\,\overline{E(g,\mathcal{B})}\,d\mu$$

for all $f, g \in L^2(X,\tilde{\mathcal{B}},\mu)$

b) $E(\cdot,\mathcal{B})$ is the unique map $: L^2(X,\tilde{\mathcal{B}},\mu) \to L^2(X,\mathcal{B},\mu)$ s.t.

$$\int_X E(f,\mathcal{B})g\,d\mu = \int_X fg\,d\mu \quad \forall f \in L^2(X,\tilde{\mathcal{B}},\mu) \ \forall g \in L^2(X,\mathcal{B},\mu)$$

c) Deduce that $E(\bar{f}, \mathcal{B}) = \overline{E(f, \mathcal{B})}$, $f \le g \Rightarrow E(f, \mathcal{B}) \le E(g, \mathcal{B})$

$E(hf, \mathcal{B}) = h\, E(f, \mathcal{B}) \quad \forall h \in L^\infty(X, \mathcal{B}, \mu)$ as well as

$|E(f, \mathcal{B})| \le E(|f|, \mathcal{B})$

d) $E(\cdot, \mathcal{B}) : L^p(X, \tilde{\mathcal{B}}, \mu) \to L^p(X, \tilde{\mathcal{B}}, \mu)$ is continuous

and $\|E(\cdot, \mathcal{B})\|_{p \to p} \le 1 \quad \forall\, 1 \le p \le \infty$.

<u>Hint:</u> Show this for $p = 1$ and ∞.

e) Let $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$ an increasing family of $\mathcal{B}$'s, and let $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ be the σ-algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{B}_n$.

Show $E(f, \mathcal{B}_n) \xrightarrow{n \to \infty} E(f, \bigvee_{n=1}^{\infty} \mathcal{B}_n)$ in L^p, $1 \le p < \infty$, for all $f \in L^p(X, \tilde{\mathcal{B}}, \mu)$.

<u>Hint:</u> See Tao, Chapter 2, Prop 3.5, if necessary.

f) Let $X = \mathbb{R}$, $\tilde{\mathcal{B}}$ = Borel σ-algebra, μ = Lebesgue measure,

$\mathcal{B}_n$ = σ-algebra generated by $\{[\frac{k}{2^n}, \frac{k+1}{2^n})\}_{k \in \mathbb{Z}}$.

Show $E(f, \mathcal{B}_n)(x) = 2^n \int_{k/2^n}^{k+1/2^n} f(y)\, dy$ for $x \in [\frac{k}{2^n}, \frac{k+1}{2^n})$

and that $\bigvee_{n=1}^{\infty} \mathcal{B}_n = \tilde{\mathcal{B}}$.

(4) Understand the one-dimensional Rising-sun Lemma (4.1 and 4.2 in Chapter 3 of Tao's notes).