Exercises 3 – Integral operators and Inequalities

(1) For $\varphi \in L^1(-1,1)$ let $(F\varphi)(\lambda) := \int_{-1}^{1} e^{i\lambda x}\varphi(x)\,dx$.

a) If $\varphi \in C_c^\infty(-1,1)$, show that $|F\varphi(\lambda)| \leq C_k(1+|\lambda|)^{-k}$ $\forall k \in \mathbb{N}$.

b) If φ is the restriction of a function $\in C^\infty(\mathbb{R})$, show that

$$\left| F\varphi(\lambda) - \frac{e^{i\lambda}}{i\lambda}\varphi(1) - \frac{e^{-i\lambda}}{i\lambda}\varphi(-1) \right| \leq C\,\lambda^{-2} \quad \text{for } |\lambda| \geq 1.$$

Can you find an approximation $\sum_{k=1}^{N} \frac{f(\varphi)}{\lambda^k}$ to $F\varphi(\lambda)$, which is correct up to an error $C\lambda^{-N-1}$?

Hint: Integration by parts!

Remark: Kim will discuss related issues later in this course.

(2) Let $\Omega = \{(x,y) \in \mathbb{R}^2 : \operatorname{Re} e^{-ixy} > 0\}$. Show that the integral operator associated to the kernel $k(x,y) = e^{-ixy}\mathbb{1}_\Omega(x,y)$ is a truncation of the Fourier transform, which is not bounded on $L^2(\mathbb{R})$.

Hint: Show that $\frac{\operatorname{Re}\langle T_k \mathbb{1}_I, \mathbb{1}_I\rangle_{L^2(\mathbb{R})}}{\langle \mathbb{1}_I, \mathbb{1}_I\rangle_{L^2(\mathbb{R})}}$ is large for large intervals $I \subset \mathbb{R}$.

(3) a) Let $K : (0,\infty)^2 \to \mathbb{C}$ be a measurable function satisfying $K(\lambda x, \lambda y) = \lambda^{-1} K(x,y)$ for $\lambda > 0$ and $\underbrace{\int_0^\infty |K(x,1)|\, x^{-1/p} dx}_{=: C} < \infty$.

Show that the operator $T_K f(y) = \int_0^\infty K(x,y)\, f(x)\, dx$ is a bounded operator on $L^p(0,\infty)$ and $\|T_K\|_{p\to p} \leq C$.

Hint: Write $T_K f(y) = \int_0^\infty K(x,1)\, f(xy)\, dx$ and use the triangle inequality for L^p-norms (Minkowski's inequality).

b) Use a) to show that the "Hilbert integral" given by $K(x,y) = \frac{1}{x+y}$ defines a continuous operator on $L^p(0,\infty)$.

c) Use a) to deduce Hardy's inequalities

$$\left(\int_0^\infty dx \left(\int_0^x dy f(y)\right)^p x^{-r-1}\right)^{1/p} \leq \frac{p}{r}\left(\int_0^\infty dy\, (y f(y))^p y^{-r-1}\right)^{1/p}$$

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where $f \geq 0$, $p \geq 1$, $r > 0$.

(4) Slightly more sophisticated than (3) and Ex. 2.2 is the weak-type Schur test (use Marcinkiewicz instead of Riesz-Thorin): If $k : X \times Y \to \mathbb{C}$ is measurable and $\|k(x,y)\|_{L^{q_0,\infty}(Y)} \leq B_0$, for a.e. $x \in X$, $\|k(x,y)\|_{L^{p_1',\infty}(X)} \leq B_1$ for a.e. $y \in Y$, $p_1, q_0 \in (1,\infty)$

$\Rightarrow \forall \Theta \in (0,1)$: $T_k : L^{p_\Theta}(X) \to L^{q_\Theta}(Y)$ bounded for

$\frac{1}{p_\Theta} = 1-\Theta + \frac{\Theta}{p_1}$, $\frac{1}{q_\Theta} = \frac{1-\Theta}{q_0}$ and $\|T_k\|_{p_\Theta \to q_\Theta} \leq C_{p_1,q_0,\Theta} B_0^{1-\Theta} B_1^{\Theta}$.

Use this to show that if $f \mapsto |x|^{-\alpha} * f$ defines a bounded operator $L^p(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ for $1 < p, r < \infty$, $0 < \alpha < n$, $\frac{1}{p} + \frac{\alpha}{n} = \frac{1}{r} + 1$.