Exercises 2 — Applications of Paley-Wiener and Interpolation

① Let $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, wlog $a_{(m,0,\dots,0)} \neq 0$.

Show that the equation $P(D)u = f \in \mathcal{E}'(\mathbb{R}^n)$ has a solution $u \in \mathcal{E}'(\mathbb{R}^n)$ $\iff$ $\dfrac{\hat{f}(\xi)}{P(\xi)} \in \mathcal{O}(\mathbb{C}^n)$.

Hint: Use / show the following fact with

$$h(z) := \frac{\hat{f}(\xi_1+z, \xi_2, \dots, \xi_n)}{P(\xi_1+z, \xi_2, \dots, \xi_n)} :$$

$$\left[\begin{array}{l} \text{If } h(z) \in \mathcal{O}(\mathbb{C}),\ p(z) = p_m z^m + \dots + p_1 z + p_0 \\ \implies |p_m h(0)| \leq \max_{|z|=1} |h(z)\, p(z)|. \end{array} \right]$$

② "Schur's test"

Let (X,μ), (Y,ν) be measure spaces, $k: X\times Y \to \mathbb{C}$ measurable.

Consider the integral operator $Kf(y) := \int_X k(x,y) f(x)\, d\mu(x)$ between suitable L^p-spaces, and let $1 \leq p_1, q_0 \leq \infty$, $p_0 = 1$, $q_1 = \infty$, $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

a) If $\| k(x,y) \|_{L^{q_0}(Y)} \leq B_0$ for a.e. $x \in X$ $\implies$ $\|K\|_{L^1(X) \to L^{q_0}(Y)} \leq B_0$.

b) If $\| k(x,y) \|_{L^{p_1'}(X)} \leq B_1$ for a.e. $y \in Y$ $\implies$ $\|K\|_{L^{p_1}(X) \to L^\infty(Y)} \leq B_1$

c) Conclude that $K: L^{p_\theta}(X) \to L^{q_\theta}(Y)$ is bounded for $\frac{1}{p_\theta} = 1-\theta + \frac{\theta}{p_1}$ and $q_\theta = \frac{q_0}{1-\theta}$, $0 \leq \theta \leq 1$.

Remark: $q_0 = p_1' = 1$; $\theta = \frac{1}{2}$ $\overset{c)}{\implies}$ $K: L^2(X) \to L^2(Y)$ bdd.

③ Young's inequality

Let $1 \le p, q, r \le \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$.

Apply Schur's test to $k(x,y) = g(y-x)$ to conclude

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} .$$

Remark: The argument applies to convolutions on any group with a bi-invariant measure (and not just $\mathbb{R}^n$).

④ Let (X,μ) be a measure space, $f: X \to \mathbb{C}$ measurable.

Define the distribution function $\lambda_f : [0,\infty) \to [0,\infty]$ by $\lambda_f(t) := \mu(\{x \in X : |f(x)| \geq t\})$. For $S \subset X$ let $\mathbb{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$

a) Check that $|f(x)|^p = p \int_0^\infty \mathbb{1}_{\{|f| \geq t\}} \, t^p \frac{dt}{t}$ and

$$\|f\|^p_{L^p(X)} = p \int_0^\infty \lambda_f(t)\, t^p \frac{dt}{t} \qquad (1 \le p < \infty)$$

$$\|f\|_{L^\infty(X)} = \inf\{t \geq 0 : \lambda_f(t) = 0\}.$$

b) Show Chebychev's inequality $\lambda_f(t) \leq t^{-p} \|f\|^p_{L^p(X)}$.

c) Show that for $1 \le p \le \infty$ $\exists\, \underline{c}_p, \overline{c}_p$:

$$\underline{c}_p \|f\|_{L^p(X)} \leq \Big(\sum_{n\in\mathbb{Z}} \lambda_f(2^n)\, 2^{np}\Big)^{1/p} = \|(2^n \lambda_f(2^n)^{1/p})_{n\in\mathbb{Z}}\|_{\ell^p(\mathbb{Z})} \leq \overline{c}_p \|f\|_{L^p}$$

d) Let $L^{p,\infty}(X) := \{ f : X \to \mathbb{C} \text{ measurable} : \|f\|_{L^{p,\infty}(X)} := \sup_{t>0} t\, \lambda_f(t)^{1/p} < \infty \}$.

Use b) to show $L^p(X) \subseteq L^{p,\infty}(X)$.

If $X = \mathbb{R}^n$ with the Lebesgue measure, show that $f(x) = |x|^{-\frac{n}{p}}$ belongs to $L^{p,\infty}(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n)$, $1 \le p < \infty$.