Exercises 1 – Fourier transform

① a) Compute the Fourier transform of $f(x) = e^{-\langle Ax, x\rangle}$ for $A \in \mathbb{C}^{n\times n}$, $\mathrm{Re}\, A$ positive definite.

b) Use this (with $n=1$) to show that if $\mathcal{F}: L^p(\mathbb{R}^n) \to L^q$ is continuous, then $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark: The Riesz-Thorin theorem shows that in this case $\mathcal{F}$ is actually continuous.

② Prove the Paley-Wiener characterization of $\mathcal{F}\,\mathcal{E}'_{B_0(R)}(\mathbb{R}^n)$ stated in the lecture (see the section from Hörmander).

③ a) Recall the proof of $\mathcal{F} L^1(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$.

b) Give 2 proofs that $\mathcal{F} L^1 \neq C_0$:

1.) Wlog $n=1$. Assume $f \in L^1(\mathbb{R})$ and $\hat{f}$ odd.

- Show $\exists A$: $\left| \int_1^b \frac{\hat{f}(x)}{x}\, dx \right| \leq A \quad \forall b < \infty$.

  (Hint: $\left| \int_\alpha^\beta \frac{\sin(x)}{x}\, dx \right| \leq B \quad \forall\, 0 \leq |\alpha| \leq |\beta| \leq \infty$)

- Show that $g(x) := \frac{\tanh(x)}{\log(1+|x|)} \in C_0(\mathbb{R}) \setminus \mathcal{F} L^1(\mathbb{R})$.

2.) It is well-known that $(L^1(\mathbb{R}^n), *)$ is a Banach algebra, not a C^*-algebra.

Check that:
- $\mathcal{F}: (L^1(\mathbb{R}^n), *) \to (C_0(\mathbb{R}^n), \cdot)$ is an injective $*$-algebra homomorphism.
- It cannot be surjective.

Remark: $\mathcal{F} L^1(\mathbb{R}^n)$ is often denoted by $A(\mathbb{R})$ and, for those following Haagerup's courses, coincides with the $A(G)$ he defines in terms of Schur multipliers.

(4) a) Show that if $f: \mathbb{R} \to \mathbb{C}$ extends to a holomorphic fct on the strip $\mathbb{R} + i(-\delta, \delta)$ for some $\delta > 0$ s.t.

$\forall |y| < \delta \ \exists C_y: |f(x+iy)| \le C_y \, \eta(x) \in L^1(\mathbb{R}_x)$

$$\Rightarrow \exists C \ \exists \varepsilon > 0 \quad |\hat{f}(\xi)| \le C e^{-\varepsilon|\xi|} \quad \forall \xi \in \mathbb{R}.$$

b) Let $A := \{ f \in C(\mathbb{R}) : \exists \delta > 0 : f \in \mathcal{O}(\mathbb{R} + i(-\delta,\delta)) \cap C(\mathbb{R} + i[-\delta,\delta])$ and $\forall N \in \mathbb{N}: \sup_{\substack{|y| < \delta \\ x \in \mathbb{R}}} |f(x+iy)| e^{N|x|} < \infty \}$

Show $\mathcal{F} A = \{ f \in \mathcal{O}(\mathbb{C}) : \forall R > 0 \ \exists \varepsilon > 0 :$

$$\sup_{\substack{|y| < R \\ x \in \mathbb{R}}} |f(x+iy)| \cdot e^{\varepsilon |x|} < \infty \}.$$

Hint: Use Cauchy's integral theorem.