① THE COMPLEX INTERPOLATION METHOD

Motivation: M_φ is defined by $M_\varphi u = \varphi(x)u(x)$. When $\varphi \in C^\infty$ and $D^\alpha \varphi \in L^\infty(\mathbb{R}^n)$ for all α, then we know from Diffund1 that $M_\varphi : H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n)$ for $k \in \mathbb{Z}$. (For $k \geq 0$, it is easily seen from the defition, and $k<0$ follows by duality).
To show that it holds for $k \in \mathbb{R}$ is very complicated, but with complex interpolation, we get it for free!

Another example: $\chi : \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism, linear outside some compact set. Define $\chi^* u(x) = u(\chi(x))$.
Then we want to conclude that $\chi^* : H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n)$ for any $k \in \mathbb{R}$.
By the chain rule it holds for integer $k \geq 0$. For $k<0$ integer:
Let $\psi = \chi^{-1}$. Then the adjoint of χ^* is $\psi^* \circ M_{|\det D\psi(x)|}$.
To conclude that it holds for $k \in \mathbb{R}$, we need interpolation

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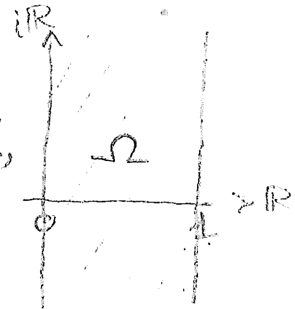
E, F Banach spaces. $F \subset E$, and $F \hookrightarrow E$ continuously.
Let $\Omega = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$.



DEF $\mathcal{H}_{E,F}(\Omega) = \{u : \Omega \to E \mid \exists C, \tilde{C}$ s.t. $\|u\|_E \leq C$ on $\bar{\Omega}$, $u(1+iy) \in F \ \forall y \in \mathbb{R}$ and $\|u(1+iy)\|_F \leq \tilde{C}\}$

That means that $u \in \mathcal{H}_{E,F}(\Omega)$ is bounded and continuous on $\bar{\Omega}$ with values in E, holomorphic on Ω with the right boundary bounded in F.

Example: $E = L^2(\mathbb{R}^n)$, $F = H^k(\mathbb{R}^n)$.

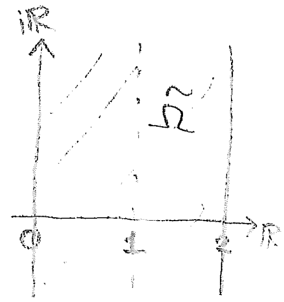
② $\boxed{\text{DEF}}$ Interpolation space : $[E,F]_\theta = \{u(\theta) \mid u \in \mathcal{H}_{E,F}(\Omega)\}$, $\theta \in [0,1]$.

Convention : $[F,E]_\theta = [E,F]_{1-\theta}$

Example : $E = \mathbb{C}$, $F = \{0\}$. Then

$\mathcal{H}_{\mathbb{C},\{0\}}(\Omega) = \{$bounded holomorphic fct. on Ω s.t. $u(1+iy) = 0$ $\forall y \in \mathbb{R}\}$.

By Schwarz' reflection principle we can extend Ω to $\tilde{\Omega} = [0,2] + i\mathbb{R}$ by setting $u(1+x+iy) = \overline{u(1+x-iy)}$ $\forall x,y \in \mathbb{R}$.



u is holomorphic on $\tilde{\Omega}$ and $u \equiv 0$ on $\{1\} \times i\mathbb{R}$,
(i.e. $u(1+iy) = 0$ $\forall y \in \mathbb{R}$).
From Complex Analysis when know that $u \equiv 0$ everywhere on $\tilde{\Omega}$.
We can therefore conclude that $\mathcal{H}_{\mathbb{C},\{0\}}(\Omega) = \{0\}$ and hence $[\mathbb{C},\{0\}]_\theta = 0$ for all $\theta \in [0,1]$. (and $[E,\{0\}]_\theta = 0$ $\forall \theta \in [0,1]$)

PROPOSITION 2.1 : E, F as above. Also $\tilde{E}, \tilde{F}$ are Banach spaces with $\tilde{F} \hookrightarrow \tilde{E}$ continuously.
Suppose $T : E \to \tilde{E}$, $F \to \tilde{F}$ continuous, linear

Then for all $\theta \in [0,1]$: $T : [E,F]_\theta \longrightarrow [\tilde{E},\tilde{F}]_\theta$.

Proof : Let $v \in [E,F]_\theta$. Show : $Tv \in [\tilde{E},\tilde{F}]_\theta$
Since $v \in [E,F]_\theta$: $\exists u \in \mathcal{H}_{E,F}(\Omega)$ st. $u(\theta) = v$ (by definition),
i.e. $\exists C, \tilde{C}$ st. $u : \Omega \to E$, $\|u\|_E \le C$ on $\bar{\Omega}$ and $u(1+iy) \in F$, $\|u(1+iy)\|_F \le \tilde{C}$

- We know that $T : E \to \tilde{E}$ cont. and lin. We want to conclude that $Tu : \Omega \to \tilde{E}$ is holomorphic.

Tu is holomorphic $\Longleftrightarrow$ Cauchy's integral thm. holds

u is assumed to be holomorphic so $u(z) = \frac{1}{\cdot} \int \ldots$

(3) We can think of the integral as an infinite sum. Then

$Tu(z) = T\frac{1}{2\pi i}\int\cdots = \frac{1}{2\pi i}\int T\cdots$, since T bounded and linear.

(We have used, that if $\varphi_n \to \varphi$ then $T\varphi_n \to T\varphi$ when T bdd. lin.).

So Tu is holomorphic.

- We also have that $\|Tu\|_{\tilde{E}} < \infty$ on $\bar{\Omega}$:

$\|Tu\|_{\tilde{E}} \leq C'\|u\|_E$, since T is bounded

$\leq C' \cdot C$, since u is assumed to be in $\mathcal{H}_{E,F}(\Omega)$

Furthermore:

- $Tu(1+iy) \in \tilde{F}$, since $u(1+iy) \in F \xrightarrow{T} \tilde{F}$.
- $\|u(1+iy)\|_{\tilde{F}} \leq C_F \|u(1+iy)\|_F$, T bdd $(F \to \tilde{F})$

$\leq C_F \cdot \tilde{C}$, since $u \in \mathcal{H}_{E,F}(\Omega)$

We have now shown that $Tu(z) \in \mathcal{H}_{\tilde{E},\tilde{F}}(\Omega)$ and hence

$Tv = Tu(\theta) \in [\tilde{E},\tilde{F}]_\theta$ by definition. $\blacksquare$

We want to identify $[H, D(A)]_\theta$, where H is a Hilbert space and $D(A)$ is the domain of A, a positive selfadjoint operator on H. From the Spectral theorem we know that there exists a unitary operator, $U: H \to L^2$ s.t. $A = U^{-1}BU$, where $Bu(x) = M_b u(x) = b(x)u(x)$, (multiplication operator) Here $D(B) = \{u \in L^2 \mid bu \in L^2\}$. We will assume that $b(x) \geq 1$.

We want to define $A^\theta = U^{-1}B^\theta U$, with $D(B^\theta) = \{u \in L^2 \mid b^\theta u \in L^2\}$.

We have $A^\theta = U^{-1}B^\theta U \Leftrightarrow A^\theta U^{-1} = U^{-1}B^\theta$. Therefore

$$D(A^\theta) = \{U^{-1}f \mid f \in D(B^\theta)\} = \{f \mid Uf \in D(B^\theta)\} = U^{-1}D(B^\theta)$$

PROPOSITION 2.2: For $\theta \in [0,1]$

$$[H, D(A)]_\theta = D(A^\theta)$$

④ Proof:

$\boxed{\text{"}\supseteq\text{"}}$ Suppose $v \in D(A^\theta)$. Show: $\exists u \in \mathcal{H}_{H,D(A)}(\Omega)$ s.t. $v = u(\theta)$, (i.e. $u: \Omega \to H$, $\|u\|_H \le C$ on $\bar{\Omega}$, $u(1+iy) \in D(A)$, $\|u(1+iy)\|_{D(A)} \le \tilde{C}$).

Define $u(z) = A^{-z+\theta} v$. Then $u(\theta) = v$. We will check that $u(z)$ does the job, ($u \in \mathcal{H}_{H,D(A)}(\Omega)$).

Note that we can write $u(z) = A^{-z+\theta} v = (\mathcal{U}^{-1} B^{-z+\theta} \mathcal{U}) v$

- $\boxed{u: \Omega \to H}$: $v \in D(A^\theta) = \mathcal{U}^{-1} D(B^\theta)$. Show $u(z) = A^{-z+\theta} v \in H$.

$$\Rightarrow A^{-z+\theta} v = \mathcal{U}^{-1} B^{-z+\theta} \mathcal{U} v = \mathcal{U}^{-1} B^{-z} \underbrace{B^\theta \underbrace{\mathcal{U}v}_{\in D(B^\theta),\text{ pr. assumption}}}_{\in L^2 \text{ (since } B^\theta : L^2 \to L^2)}$$

Applying B^{-z} to $B^\theta \mathcal{U} v \in L^2$, we want to stay in L^2. But $B^{-z}: L^2 \to L^2$ bdd, since: $\left|\frac{1}{b^z}\right| = |b|^{-\mathrm{Re} z} \le 1$, because $\mathrm{Re} z \ge 0$ and $|b| \ge 1$.

So $B^{-z} B^\theta \mathcal{U} v \in L^2 \Rightarrow \mathcal{U}^{-1} B^{-z} B^\theta \mathcal{U} v \in H$.

- $\boxed{\text{u holomorphic and bounded}}$: $\|u\|_H = \|A^{-z+\theta} v\|_H = \|\mathcal{U}^{-1} B^{-z+\theta} \mathcal{U} \cdot \mathcal{U}^{-1} b\|_H$, for some $b \in D(B^\theta)$

$$\begin{aligned}
&= \|B^{-z} B^\theta b\|_{L^2} \\
&\le \underbrace{\|B^{-z}\|_{L^2}}_{\le 1,\text{ as above.}} \cdot \underbrace{\|B^\theta b\|_{L^2}}_{<\infty,\text{ since } B^\theta : L^2 \to L^2,\text{ and } b \in D(B^\theta).} \\
&\le C \quad \forall z,
\end{aligned}$$

and hence u is bounded in H on $\bar{\Omega}$.

- $\boxed{u(1+iy) \in D(A)}$: It is equivalent to $\|A u(1+iy)\|_H < \infty$.

$$\|Au(1+iy)\|_H = \|A \cdot A^{-1-iy+\theta} v\|_H = \|A^{-iy} \underbrace{A^\theta v}_{\in H,\text{ since } v \in D(A^\theta) \to H.}\|_H$$

$$\begin{aligned}
&\le \|A^{-iy}\|_H \cdot \underbrace{\|A^\theta v\|_H}_{<\infty} \\
&\le C' \|\mathcal{U}^{-1} B^{-iy} \mathcal{U}\|_H \\
&\le C,
\end{aligned}$$

since: $\mathcal{U}^{-1} B^{-iy} \mathcal{U} : H \to H$ bdd $\Leftrightarrow$ $B^{-iy}: L^2 \to L^2$ bdd

$\left|\frac{1}{b^{iy}}\right| = |b|^{-\mathrm{Re}(iy)} = |b|^0 = 1$, so B^{-iy} is bdd.

(5) • $\boxed{\|u(1+iy)\|_{D(A)} \leq \tilde{C}}$: $\|\cdot\|_{D(A)}$ is the graph norm : $\|u\|_{D(A)} = \|u\|_H + \|Au\|_H$

So $\|u(1+iy)\|_{D(A)} = \underbrace{\|u(1+iy)\|_H}_{\substack{<\infty \\ (u \text{ bdd. on } \bar{\Omega} \text{ in } H)}} + \underbrace{\|Au(1+iy)\|_H}_{\substack{<\infty \\ (\text{just showed})}} \leq \tilde{C} < \infty$.

We have now checked that u does the job indeed and hence $v \in [H, D(A)]_\theta$.

$\boxed{\text{"}\subseteq\text{"}}$ Suppose $u(z) \in \mathcal{H}_{(H, D(A))}(\Omega)$ <u>Show</u> : $u(\theta) \in D(A^\theta)$, which is equivalent to showing $\|A^\theta u(\theta)\|_H < \infty$.

We would like to use the maximum modulus principle, since u is bounded on $\bar{\Omega}$. But Ω is not a bounded set, so maybe $A^z u(z)$ goes to infinity for some z. Instead of looking at $A^z u(z)$, we look at $A^z(1+i\varepsilon A)^{-1}u(z)$. (In the end we will take $\varepsilon \to 0$).

This is indeed a bounded operator from $\Omega \to H$:

$$A^z(1+i\varepsilon A)^{-1}u(z) = \mathcal{U}^{-1}B^z(1+i\varepsilon B)^{-1}\underbrace{\mathcal{U}u(z)}_{\substack{\text{bdd. holomorphic fct from } \Omega \to L^2, \text{ since } u \\ \text{bdd. holom. from } \Omega \to H.}}$$

Furthermore $B^z(1+i\varepsilon B)^{-1}$ is bounded :

$$\left|\frac{b^z}{1+i\varepsilon b}\right| = \frac{|b^z|}{(1+\varepsilon^2 b^2)^{1/2}} = \frac{|b|^{\mathrm{Re}\, z}}{(1+\varepsilon^2 b^2)^{1/2}} \leq \frac{|b|}{(1+\varepsilon^2 b^2)^{1/2}}, \text{ since } \mathrm{Re}\, z \in [0,1].$$

$$\leq \frac{1}{\varepsilon} = C_\varepsilon \text{ for all } \varepsilon \in \mathbb{R}.$$

So $\mathcal{U}^{-1}B^z(1+i\varepsilon B)^{-1}\mathcal{U}u(z)$ is a bounded holomorphic fct from $\Omega \to H$.

We can then use the max. principle :

$$\|A^z(1+i\varepsilon A)^{-1}u(z)\|_H \leq \sup_{y \in \mathbb{R}} \max\left\{\|A^{iy}(1+i\varepsilon A)^{-1}u(iy)\|_H, \|A^{1+iy}(1+i\varepsilon A)^{-1}u(1+iy)\|_H\right\}$$

<u>Left boundary</u> : $\|\underbrace{A^{iy}}_{\substack{\text{bdd on } H \\ (\text{shown earlier})}}(1+i\varepsilon A)^{-1}\underbrace{u(iy)}_{\in H}\|_H$

We need $\|(1+i\varepsilon A)^{-1}\|_H$ to be bounded independently of ε.

$\|(1+i\varepsilon A)^{-1}\| = \|(1+i\varepsilon B)^{-1}\|$

⑥ $\left|\frac{1}{1+i\varepsilon b}\right| = \frac{1}{1+\varepsilon^2 b^2} \leq 1$ for all $\varepsilon \in \mathbb{R}$, since $a \geq b \geq 1$, i.e. $\|(1+i\varepsilon A)^{-1}\|_H$ bdd indep. of ε, and hence $\|A^{iy}(1+i\varepsilon A)^{-1}u(iy)\|_H \leq \sup_{y\in\mathbb{R}}\{C\,\|u(iy)\|_H\}$

Right boundary: $\|A^{1+iy}(1+i\varepsilon A)^{-1}u(1+iy)\|_H$

$= \|\underbrace{A^{iy}(1+i\varepsilon A)^{-1}}_{\substack{\text{bdd on } H \text{ indep. of } \varepsilon \\ \text{(just showed)}}} A u(1+iy)\|_H \leq C\|\underbrace{Au(1+iy)}_{\in D(A)\text{, since } u \in \mathcal{H}(H, D(A))(\Omega)}\|_H$

$\leq \|u(1+iy)\|_{D(A)}$, $(\|u\|_{D(A)} = \|u\|_H + \|Au\|_H)$

So: $\|A^z(1+i\varepsilon A)^{-1}u(z)\|_H \leq \sup_{y\in\mathbb{R}} \max\{C\|u(iy)\|_H, C\cdot\|u(1+iy)\|_{D(A)}\}$

$\leq G$ (independent of ε).

Let $\varepsilon \to 0$: $\|A^z u(z)\|_H \leq G$. for $z = \theta$: $\|A^\theta u(\theta)\|_H \leq G < \infty$,

so $u(\theta) \in D(A^\theta)$

Application

Recall: $H^s(\mathbb{R}^n) = \{u \in S' \mid \langle\xi\rangle^s \hat{u}(\xi) \in L^2\}$, $s \in \mathbb{R}$

$= \{u \in S' \mid \mathcal{F}^{-1}\langle\xi\rangle^s \hat{u}(\xi) \in L^2\}$, ($\mathcal{F}^{-1}: L^2 \to L^2$ isometri).

From Dif.fund: $\mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F} = \Lambda^s$, where $\Lambda = (1-\Delta)^{1/2}$.

$\Lambda^s : S' \to L^2$ and is selfadjoint, $(s \geq 0)$

$= \{u \in S' \mid \Lambda^s u \in L^2\}$

$\overset{s\geq 0}{=} D(\underbrace{\Lambda^s_{max}}_{=\Lambda^s_{min} \text{ (from Dif fund) on } \mathbb{R}^n})$, by definition

$= D(\Lambda^s)$

So for $s \geq 0$: $H^s(\mathbb{R}^n) = D(\Lambda^s)$

Applying Prop. 2.2 with $H = L^2(\mathbb{R}^n)$, $A = \Lambda^s$ $(s\geq 0)$, we get

$$\boxed{[L^2(\mathbb{R}^n), H^s(\mathbb{R}^n)]_\theta = H^{s\theta}(\mathbb{R}^n) \qquad \theta \in [0,1]}$$

In fact we can generalize: For all $\sigma, t \in \mathbb{R}$:

$[H^\sigma(\mathbb{R}^n), H^t(\mathbb{R}^n)] = H^{\theta t + (1-\theta)\sigma}(\mathbb{R}^n) \qquad \theta \in [0,1]$

7 We see that from the following:

In Diffund we showed that $\Lambda^s = (1-\Delta)^{s/2}$ is an isometri from H^t to H^{t-s} for all $t \in \mathbb{R}$ (and $s \in \mathbb{R}$).

We must then have that
$$\Lambda^s [E,F]_\theta = [\Lambda^s E, \Lambda^s F]_\theta .$$
$$\{\Lambda^s u(\theta)\} = \{(\Lambda^s u)(\theta)\}$$

We are looking at $E = L^2(\mathbb{R}^n) = L^2$ and $F = H^k(\mathbb{R}^n) = H^k$ (I will leave out $(\mathbb{R}^n)$ in the following in my notation).

$\Lambda^s L^2 = H^{-s}$, and $\Lambda^s H^k = H^{k-s}$.

$$[H^{-s}, H^{k-s}]_\theta = [\Lambda^s L^2, \Lambda^s H^k]_\theta = \Lambda^s [L^2, H^k]_\theta = \Lambda^s H^{k\theta} = H^{k\theta - s}.$$

If we set $\sigma = -s$, we have: $[H^\sigma, H^{k+\sigma}]_\theta = H^{k\theta+\sigma}$

and set $t = k+\sigma$: $[H^\sigma, H^t]_\theta = H^{(t-\sigma)\theta+\sigma} = H^{t\theta+\sigma(1-\theta)}$.

We now return to the introduction and want to conclude that

(*) $M_\varphi : H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n) \quad \forall k \in \mathbb{R}$.

For positive k, we can express k as $k = s\theta$, where $s \in \mathbb{N}_0$ and $\theta \in [0,1]$.
Then $H^k(\mathbb{R}^n) = H^{s\theta}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), H^s(\mathbb{R}^n)]_\theta$
Since we know that (*) holds for integer k's, $M_\varphi : L^2 \to L^2$ and $M_\varphi\ H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, and hence prop. 2.1 gives us what we want.

If $k<0$ and $k \notin \mathbb{Z}_-$: $\exists\, t \in \mathbb{Z}_+$ s.t. $-t > k > -(t+1)$.
$$[H^{-t}, H^{-(t+1)}]_\theta = H^{-\theta t - (1-\theta)(t+1)} = H^{-(t+1)+\theta} = H^k, \text{ for some } \theta \in [0,1].$$
Using prop. 2.1, we get the result.

In the same way with χ^*.

If $F \not\subset E$ (and $E \not\subset F$), we want to define the interpolation space.
Suppose $E \hookrightarrow V$, $F \hookrightarrow V$ continuously, V (Banach space.)
Let $G = \{e+f \mid e \in E, f \in F\}$ - (also Banach space.)

(8) $\boxed{\text{DEF}}$ $\mathcal{H}_{E,F}(\Omega) = \{u(z)$ bounded and continuous in $\bar{\Omega}$, $u : \Omega \to G$, holomorphic in Ω : $\|u(iy)\|_E$ and $\|u(1+iy)\|_F$ bounded for all $y \in \mathbb{R}\}$.

$\boxed{\text{DEF}}$ $[E,F]_\theta$ as before.

PROPOSITION 2.3: $0<\theta<1$: $\left[[L^{p_1}(\mathfrak{X},\mu), L^{p_2}(\mathfrak{X},\mu)]_\theta = L^q(\mathfrak{X},\mu) \right]$,
($(\mathfrak{X},\mu)$ not nessesarily finite or a Lebesgue-measure.)

where p_1, p_2 and q are related by $\frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$

Remark: Prop. 2.1 + Prop. 2.3 gives us Riesz-Thorin.

$$\left.\begin{array}{l} T : L^{p_1} \to L^{\tilde{p}_1} \text{ bdd} \\ T : L^{p_2} \to L^{\tilde{p}_2} \text{ bdd} \end{array}\right\} \Rightarrow T : \underbrace{[L^{p_1}, L^{p_2}]_\theta}_{= L^q} \longrightarrow \underbrace{[L^{\tilde{p}_1}, L^{\tilde{p}_2}]}_{= L^{\tilde{q}}}$$

We can also use interpolation to define Sobolev space on domains: If $\Omega \subseteq \mathbb{R}^n$

$s \in \mathbb{R}, s \geq 0$: $H^s(\Omega) := [L^2(\Omega), H^k(\Omega)]_\theta$, $k \geq s$, $s = \theta k$

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Litterature: Michael E. Taylor; "Partial Differential Equations", Vol. I.