Adaptive finite elements for nonlocal problems

Error estimates and mesh refinements for friction, contact, cracks

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Prediction and Data Assimilation for Nonlocal Diffusions
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Contents

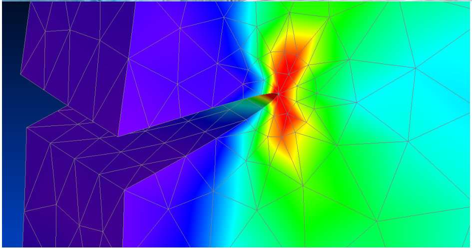
- Nonlocal operators with contact and in singular domains
- Example: Adaptivity and mesh refinements for nonlinear fractional contact problems
- Adaptivity: waves + elasticity + friction
  (for today no coupling to local problems)

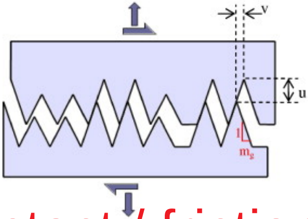
HG, J. Stocek, Space-time adaptive finite elements for nonlocal parabolic variational inequalities, in preparation.

HG, F. Meyer, C. Oezdemir, D. Stark, E. P. Stephan, Boundary elements with mesh refinements for the wave equation, Numer. Math. (2018).

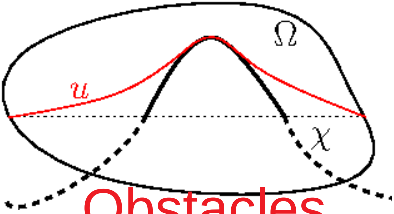
L. Banz, HG, A. Issaoui, E. P. Stephan, Stabilized mixed hp-BEM for frictional contact problems in linear elasticity, Numer. Math. (2017).

See also http://www.macs.hw.ac.uk/∼hg94.





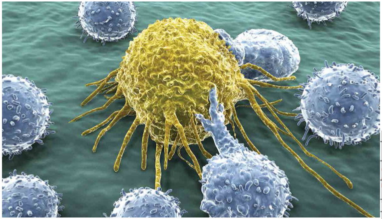
Contact / friction



Obstacles

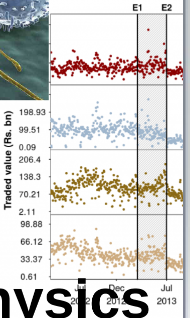
nonsmooth solutions:
adaptive methods for nonlocal operators

Nonlocal operators: recent and classical



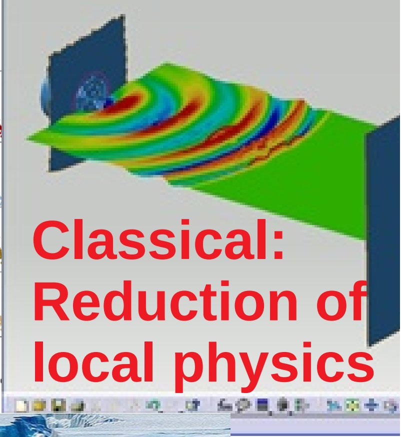
Estrada, HG, Painter, SIAP 2018

Recent: Nonlocal physics

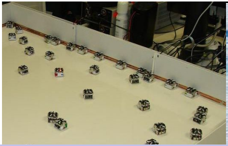


Estrada, HG, draft 2018

Banz, HG, Nezhi, Stephan, Comput. Mech. 2016

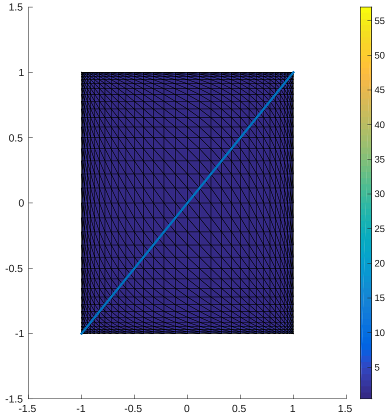


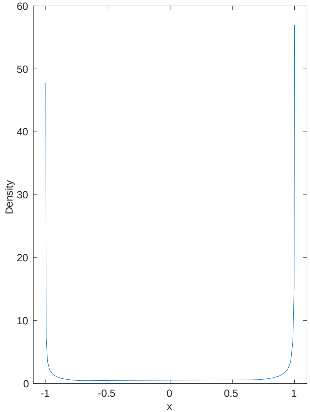
Classical: Reduction of local physics



Wave scattering off a crack: singular stresses

Crack problem $\sim -(-\Delta)^{-1/2}u = f$
$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $f = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$.
solution near corner $r^{-0.703\ldots}$, solution near edge $r^{-0.5\ldots}$

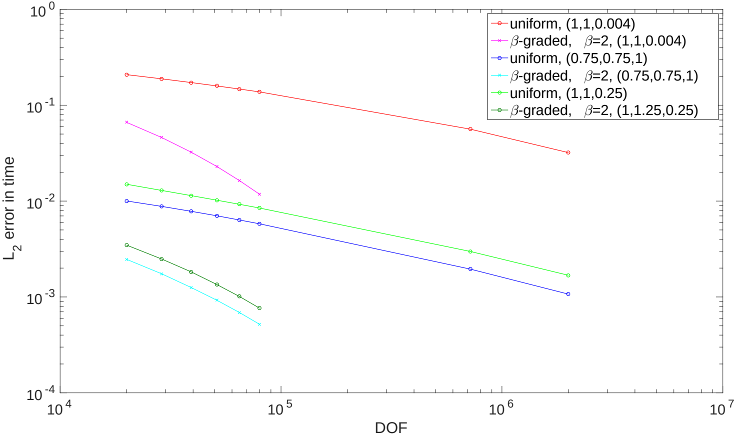




Wave scattering off a crack: error of numerical solutions

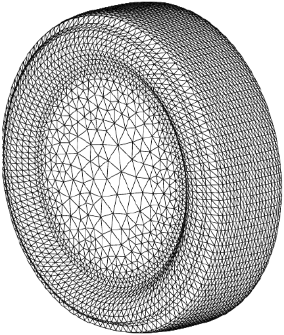
Crack problem $\sim -(-\Delta)^{-1/2}u = f$
$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $f = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 0.5$.

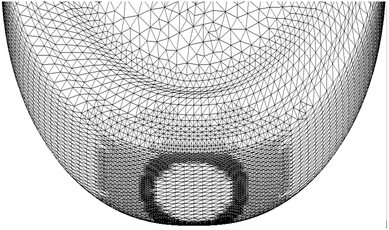
error $\|u(x_0,t) - u_{h,\Delta t}(x_0,t)\|_{L^2_t}$: uniform vs. graded meshes



HG, Meyer, Özdemir, Stark, Stephan, Numerische Mathematik 2018.

Adapted meshes for real problems





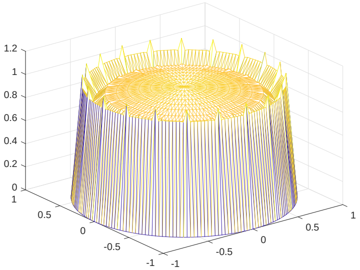
Steady state of fractional diffusion

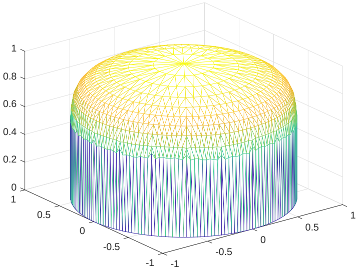
$$-(-\Delta)^{1/10}u = 1$$

$\Omega = \{|x| < 1\}$.

Exact solution: $u(x) = (1 - |x|^2)_+^{1/10}$

Uniform vs. graded mesh





Nonlocal diffusion

Heat equation with fractional Laplacians:

$$\partial_t u + (-\Delta)^s u(x) = \partial_t u + c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathsf{d}y = f(x), \qquad s \in (0,1)$$

Nonlocal model problems of recent interest:

Numerical analysis: Ainsworth, Nochetto, Otarola, Salgado, ...

Nonlinear PDE: Caffarelli, Figalli, Grubb, Ros-Oton, ...

Image processing, Financial math, Math biology / ecology, Levy robotics, ...

This workshop: Nonlocal operators from ice dynamics.

Fractional Laplacian

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y, \qquad s \in (0,1) \ .$$

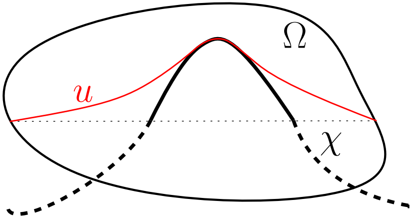
- Careful: This is the generator of the Lévy process and has physical meaning. Used in probability, PDE, applications.
- Recent numerical analysis of $(-\Delta)^s$: Ainsworth, Glusa (2017), Nochetto et al. (2017)
- Don't confuse with spectral $(-\Delta)^s$, the fractional power of the Dirichlet problem.
  Numerical analysis by Otarola, Nochetto and many others (2015 –)
  "Dirichlet-Neumann operator" for degenerate Laplace eqn. on $\mathbb{R}^+ \times \Omega$.

Fractional contact problems

Heat equation with fractional Laplacian:

$$\partial_t u + (-\Delta)^s u(x) = \partial_t u + c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathrm{d}y = f(x), \qquad s \in (0,1)$$

Obstacle problem: (see also Nochetto et al., Otarola–Salgado ...)



we actually consider general contact conditions (non-penetration, friction) of relevance from nonlocal materials to image processing
in preparation with J. Stocek

Fractional contact problems

Heat equation with fractional Laplacian:

$$\partial_t u + (-\Delta)^s u(x) = \partial_t u + c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathrm{d}y = f(x), \qquad s \in (0,1)$$

we actually consider general contact conditions (non-penetration, friction) of relevance from nonlocal materials to image processing

Numerical analysis: Otarola and Salgado (SINUM 2016)
Nonlinear PDE: Figalli and Caffarelli (J. Reine Angew. Math. 2013), Begonas, Figalli, Ros-Oton (Inventiones 2017), ...
Imaging: Osher, Schönlieb, ...
Financial math: Merton (J. Finan. Econ. 1976)
Mathematical Biology: Carrillo (2018)

Variational formulation

$$
\begin{aligned}
-(-\Delta)^s u &= f && \text{in } \Omega \\
u &= 0 && \text{in } \mathbb{R}^n \setminus \overline{\Omega}
\end{aligned}
$$

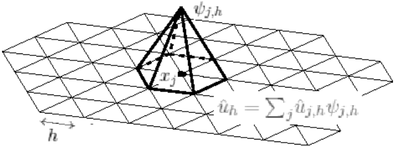
Find $u \in H_0^s(\Omega)$ such that for all $v \in H_0^s(\Omega)$

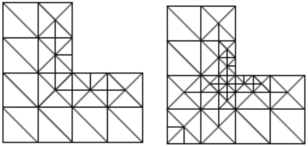
$$
\frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dy\ dx = \int_\Omega v(x) f(x) dx
$$

unique solution

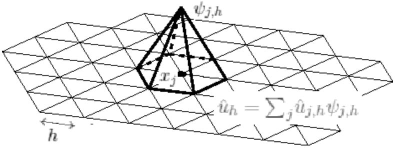
Discretization

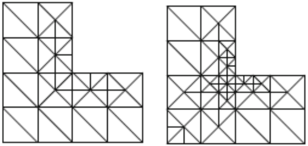
- $T = \cup_{i=1}^{M} T_i$ triangulation (quasi-uniform or graded)
- V_h piecewise polynomial functions of degree p on $\Gamma = \cup_{i=1}^{M} T_i$ (continuous if $p \geq 1$)
- usually $p = 1$, $\widetilde{V}_h$ subspace vanishing at boundary





Discretization





Variational formulation: Find $u_h \in \widetilde{V}_h$ such that for all $v_h \in \widetilde{V}_h$

$$\frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_h(x) - u_h(y))(v_h(x) - v_h(y))}{|x-y|^{n+2s}} dy\ dx = \int_\Omega v_h(x) f(x) dx$$

unique solution

For the heat equation $\partial_t u + (-\Delta)^s u = f$: $dG(q)$ in time ($q = 0$ implicit Euler).

Singularities and graded meshes

$$
\begin{aligned}
-(-\Delta)^s u &= f && \text{in } \Omega \\
u &= 0 && \text{in } \mathbb{R}^n \setminus \overline{\Omega}
\end{aligned}
$$

Theorem (folklore, Ros-Oton–Serra '14, Grubb '15)

Let $s \neq \frac{1}{2}$, Ω suff. smooth, $f \in L^\infty(\Omega)$.
Then $\frac{u(x)}{\mathrm{dist}(x,\partial\Omega)^s} \in C^\alpha(\Omega)$ for some $\alpha > 0$.
Logarithmic corrections for $s = \frac{1}{2}$.

Corollary

Quasi-optimal convergence on β-graded meshes:

$$\|u - u_h\|_{H^s} \lesssim h^{\min\{\beta s, \frac{3}{2}\} - \varepsilon} \ .$$

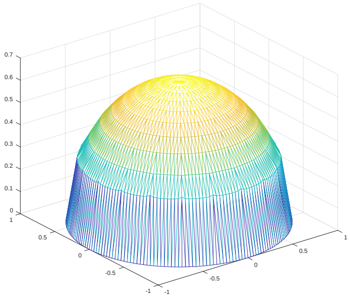
radial nodes of β-graded mesh on unit disk: $r_j = 1 - \left(\frac{j}{N}\right)^\beta$, $j = 1, \ldots, N$.

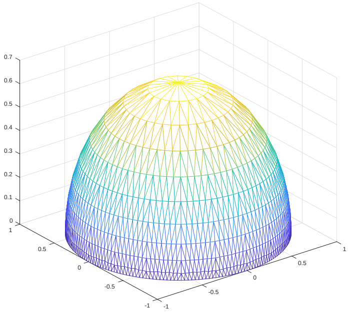
Numerical experiment: $-(-\Delta)^s u = f$

$\Omega = \{|x| < 1\}$. $f = 1$, $s = \frac{1}{2}$

Exact solution: $u(x) = (1 - |x|^2)_+^s$

Uniform vs. graded mesh



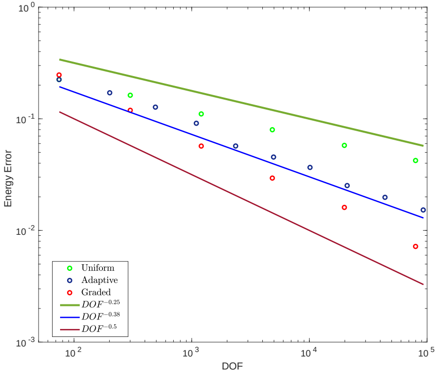


Convergence: $-(-\Delta)^s u = f$

$$\begin{cases} (-\Delta)^s \quad u = 1 \text{ in } \Omega = B_1 \\ \qquad\quad\;\; u = 0 \text{ in } \Omega^c. \end{cases}$$

Exact solution: $u(x) = (1-|x|^2)_+^s$

Convergence rates for $s = \frac{1}{2}$
$\frac{1}{2}$ (uniform), 1 (graded)

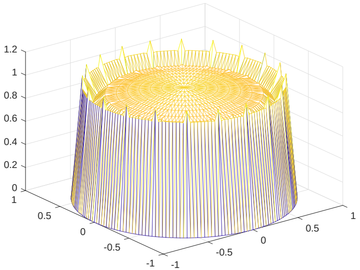


Numerical experiment: $-(-\Delta)^s u = f$

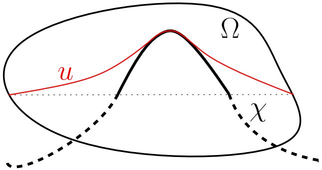
$\Omega = \{|x| < 1\}$. $f = 1$, $s = \frac{1}{10}$

Exact solution: $u(x) = (1 - |x|^2)_+^s$

Uniform vs. graded mesh



Time-independent Obstacle Problem



Let $f \in H^{-s}(\Omega)$, $\chi \in V_h$, $\chi \leq 0$ on $\partial\Omega$.
Find $u \in H_0^s(\Omega)$, $u \geq \chi$, such that for all $v \in H_0^s(\Omega)$ with $v \geq \chi$

$$\langle (-\Delta)^s u, v - u \rangle_{H^{-s},H^s} \geq \langle f, v - u \rangle_{H^{-s},H^s}.$$

$$a(u, v - u) \geq f(v - u),$$

Discretized weak form: Find $u_h \in \widetilde{V}_h$, $u \geq \chi_h$, such that for all $v_h \in \widetilde{V}_h$ with $v_h \geq \chi_h$

$$a(u_h, v_h - u_h) \geq f(v_h - u_h),$$

Computable error estimates (time-independent)

Theorem

Let $u_h \in V_h$ be the discrete solution of problem and $U \in H^s$ be the solution of the associated linear problem. Then

$$\|u - u_h\|_{H^s} \leq \|U - u_h\|_{H^s} + \|\sigma_h - \sigma_+\|_{H^{-s}} + \langle \sigma_+, u_h - \chi \rangle^{\frac{1}{2}}.$$

Using the local a posteriori estimator:
(Classical)

$$\|u - u_h\|_{H^s} \lesssim \left(\sqrt{\sum_{T \in S_z} h^{2s} \|\sigma_h\|^2_{L^2(T)}} + \|\sigma_h - \sigma_+\|_{H^{-s}} + \langle \sigma_+, u_h - \chi \rangle^{\frac{1}{2}} \right).$$

(Nochetto et al 2010)

$$\|u - u_h\|_{H^s} \lesssim \left(\sum_{z \in \mathcal{P}_h \setminus \mathcal{C}_h} h^{2s} \|(\sigma_h - \overline{\sigma_h}) \Lambda_z\|_{L^2(S_z)} - \sum_{z \in \mathcal{F}_h} \overline{\sigma_h} \langle u_h - \chi, \Lambda_z \rangle \right)^{\frac{1}{2}}$$

Computable error estimates (time-independent)

Using the local a posteriori estimator:
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$$\|u-u_h\|_{H^s} \lesssim \left(\sqrt{\sum_{T\in S_z} h^{2s}\|\sigma_h\|^2_{L^2(T)}} + \|\sigma_h - \sigma_+\|_{H^{-s}} + \langle \sigma_+, u_h - \chi\rangle^{\frac{1}{2}} \right).$$

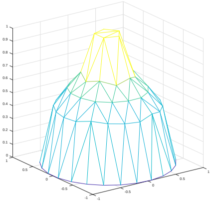
(Nochetto et al 2010)

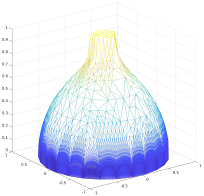
$$\|u-u_h\|_{H^s} \lesssim \left(\sum_{z\in\mathcal{P}_h\setminus\mathcal{C}_h} h^{2s}\|(\sigma_h - \overline{\sigma_h})\Lambda_z\|_{L^2(S_z)} - \sum_{z\in\mathcal{F}_h} \overline{\sigma_h}\langle u_h - \chi, \Lambda_z\rangle \right)^{\frac{1}{2}}$$

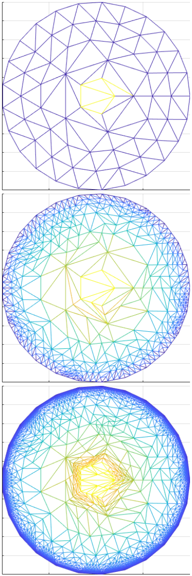
Note that RHS is computable. This can be used as an error indicator for adaptive algorithm.

Adaptive algorithm in practice

1. Start with coarse grid: $(\Delta x)_i = h_0 \; \forall \Delta_i$
2. Solve Finite Element problem
3. Compute error indicator $\eta(\Delta_i)$
4. $\eta(\Delta_i) > \delta \eta_{max} \quad \Longrightarrow \quad$ refine
5. GO TO 2.







Computable error estimates (time-dependent)

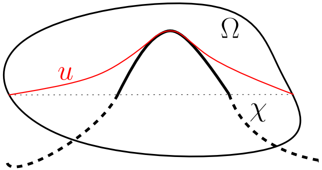
Residual:

$$\sigma_h^k = \delta u_h^k + (-\Delta)^s u_h^k - f^k$$

Theorem

$$\begin{aligned}
&\|u(t_N) - u_h^N\|^2_{L^2(\Omega)} \\
&\quad + \int_0^T \|u - \overline{u_h}\|^2_{H^s} + \|u - u_h\|^2_{H^s} + \|\partial_t(u - u_h) + (\sigma - \sigma_h)\|^2_{H^{-s}} \mathrm{d}t \\
&\quad \lesssim \|u_0 - u_h^0\|^2_{L^2(\Omega)} + \sum_k \tau_k \|u_h^k - u_h^{k-1}\|^2_{H^s} \\
&\quad + \sum_k \tau_k \left[\sum_{z \in \mathcal{P}_h} h^{2s} \|(\sigma_h^k - \overline{\sigma_h^k})\Lambda_z\|^2_{L^2(S_z)} - \sum_{z \in \mathcal{F}_h} \overline{\sigma_h} \langle u_h^k - \xi, \Lambda_z \rangle \right]
\end{aligned}$$

Time-dependent Obstacle Problem



Let $f \in L^2(0,T;H^{-s}(\Omega))$, $\chi \in V_{h,\tau}$, $\chi \leq 0$ on $\partial\Omega$.
Find $u \in L^2(,0,T;H_0^s(\Omega))$, $u \geq \chi$, such that for all $v \in L^2(0,T;H_0^s(\Omega))$ with $v \geq \chi$

$$\langle \partial_t u, v-u\rangle + \langle (-\Delta)^s u, v-u\rangle_{H^{-s},H^s} \geq \langle f, v-u\rangle_{H^{-s},H^s}.$$

$$\langle \partial_t u, v-u\rangle + a(u,v-u) \geq f(v-u),$$

Discretized weak form: Find $u_h \in \widetilde{V}_{h,\tau}$, $u \geq \chi_h$, such that for all $v_h \in \widetilde{V}_{h,\tau}$ with $v_h \geq \chi_h$

$$\langle \partial_t u_h, v_h-u_h\rangle + a(u_h,v_h-u_h) \geq f(v_h-u_h),$$

Mixed Formulation

$$
\begin{aligned}
(-\Delta)^s u - f &\geq 0, \\
u - \chi &\geq 0, \\
((-\Delta)^s u - f)(u - \chi) &= 0, \text{a.e.} \,\text{on}\, \Omega.
\end{aligned}
$$

Let $h \in H^{-s}(\Omega)$.
Find $(u, \lambda) \in H_0^s(\Omega) \times H^{-s}(\Omega)^+$ such that

$$
\begin{cases}
a(u, v) + b(\lambda, v) = \langle f, v \rangle \\
b(\mu - \lambda, u - \chi) \leq 0,
\end{cases}
$$

for all $(v, \mu) \in H_0^s(\Omega) \times H^{-s}(\Omega)^+$.
The mixed formulation is equivalent to the original variational inequality.

Computable error estimates – time-independent obstacle problem

Theorem

Let $(u, \lambda), (u_h, \lambda_h)$ *be solutions of the continuous and discrete problem, respectively. Let* $R := f - (-\Delta)^s u_h$. *Then*

$$\|u - u_h\|_{H^s}^2 + \|\lambda - \lambda_h\|_{H^{-s}}^2 \lesssim \sum_{z \in \mathcal{P}_h \setminus \mathcal{C}_h} h^{2s+d-\frac{2d}{p}} \|R - R_z\|_{L^p(S_z)}^2 + \sum_{z \in \mathcal{F}_h} s_z d_z,$$

where $d_z = \langle u_h - \chi, \phi_z \rangle$, *and* $s_z = \frac{\langle R, \phi_z \rangle}{\langle 1, \phi_z \rangle}$ *for* $z \in \mathcal{P}_h \cap \Omega$.

Adaptive algorithm in practice

1. Start with coarse grid: $(\Delta x)_i = h_0 \ \forall \Delta_i$
2. Solve Finite Element problem
3. Compute error indicator $\eta(\Delta_i)$
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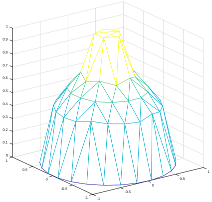
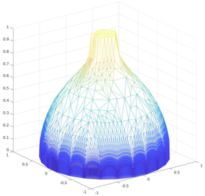
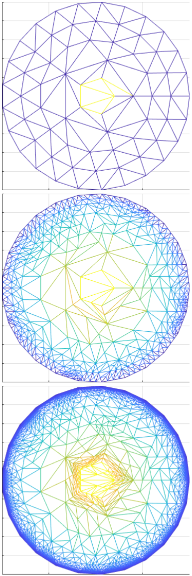
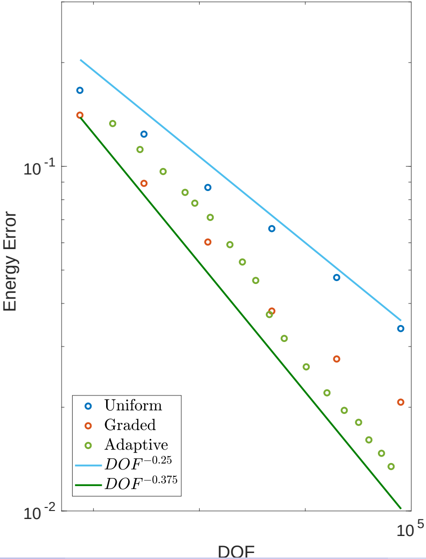






Convergence of adaptive method



Computable error estimates – time-dependent obstacle problem

Theorem

Let $(u,\lambda),(u_h,\lambda_h)$ *be solutions of the continuous and discrete problem, respectively. Let* $R := f-(-\Delta)^s u_h - \partial_t u_h$. *Then*

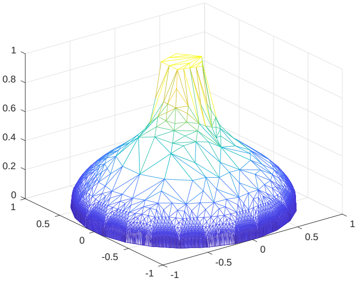
$$\|(u-u_h)(T)\|^2_{L^2(\Omega)} + \int_0^T \|u-u_h\|^2_{H^s(\Omega)} + \|(\lambda-\lambda_h)\|^2_{H^{-s}(\Omega)}\mathrm{d}t + \sum_{k=1}^{M}$$

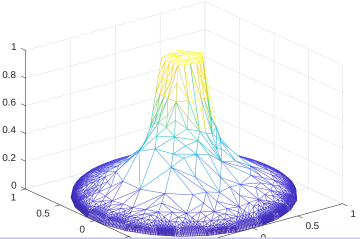
$$\lesssim \|u_0-u_h^0\|^2_{L^2(\Omega)} + \sum_{k=1}^{M}\tau_k\|u_h^k-u_h^{k-1}\|^2_{\mathbb{V}}$$

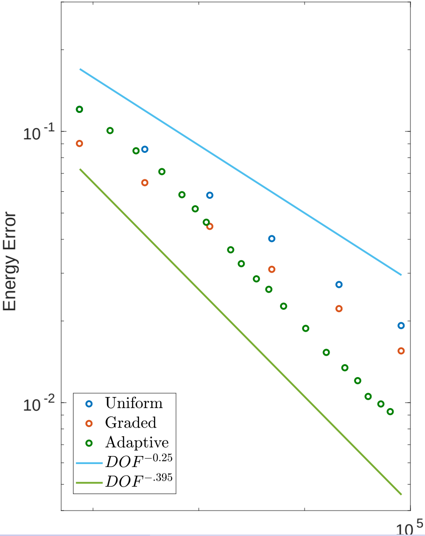
$$+\sum_{k=1}^{M}\tau_k\left[\sum_{z\in\mathcal{P}_h} h^{2s}\|(R^k-R^k_z)\phi_z\|^2_{L^p(S_z)} - \sum_{z\in\mathcal{F}_h} s^k_z\langle u^k_h-\chi,\phi\right.$$

where s^k_z *is defined as for the elliptic obstacle problem.*

Adaptivity: time-dependent obstacle problem







BEM for Δ: Green's function G / single-layer ansatz

Fundamental solution $G(x)$ = electric potential of a point charge:

$$\mathbb{R}^2: \quad -\Delta\left(\frac{1}{2\pi}\log(|x|)\right) = \delta(x) \quad , \qquad \mathbb{R}^3: \quad -\Delta\left(\frac{1}{4\pi|x|}\right) = \delta(x) \ .$$

Single layer ansatz / potential

$$u(x) = \mathcal{S}\phi(x) := \int_\Gamma G(x-y)\ \phi(y)\ ds_y, \quad x \in \mathbb{R}^d \setminus \Gamma,$$

is continuous and solves Laplace equation on $\mathbb{R}^d \setminus \Gamma$: $-\Delta u = 0$.

Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = f \text{ on } \Gamma = \partial\Omega$$

$$\Longleftrightarrow \qquad \mathcal{V}\phi(x) := \int_\Gamma G(x-y)\ \phi(y)\ ds_y = f(x)\ , \qquad x \in \Gamma$$

Note $\mathcal{V} \simeq (-\Delta)^{-1/2}$. $\mathcal{W} \simeq (-\Delta)^{1/2}$. Many numerical methods adapt.

BEM for Δ: computable error estimates

Dirichlet problem for $u(x) = \int_\Gamma G(x-y)\ \phi(y)\ ds_y$

$$\Delta u = 0 \text{ in } \Omega, \quad u = f \text{ on } \Gamma = \partial\Omega$$

$$\iff \qquad \mathcal{V}\phi(x) := \int_\Gamma G(x-y)\ \phi(y)\ ds_y = f(x)\ , \qquad x \in \Gamma$$

Finite element solution ϕ_h

$$\langle \mathcal{V}\phi_h, \psi_h \rangle = \langle f, \psi_h \rangle \quad \text{for all } \psi_h \in H_h \subset H^{-\frac{1}{2}}(\Gamma)$$

BEM for Δ: computable error estimates

Theorem (Carstensen–Stephan '95, Carstensen '96, 2d)

- Ω polygonal domain, f continuous and smooth on each side of Γ
- quasi-uniform triangulation of Γ, pw. constant ansatz functions
- $\mathcal{R}_h = f - \mathcal{V}\phi_h$

$$\Longrightarrow \ \forall s \in [0,1] \ \forall 0 < h < h_0 : \ \|\phi - \phi_h\|_{H^{-s}(\Gamma)} \simeq h^s \|\partial_\Gamma \mathcal{R}_h\|_{L^2(\Gamma)}$$

BEM for waves: computable error estimates

Theorem

Let $\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that
$\mathcal{R} = \partial_t\ f - \mathcal{V}\partial_t \phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^1(\Gamma)) \Longrightarrow$

$$\|\phi - \phi_{h,\Delta t}\|^2_{0,-\frac{1}{2}} \lesssim \sum_{i,\Delta} \max\{\Delta t, h_\Delta\}\ \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

$$\max\{\Delta t, h\}\|\mathcal{R}\|^2_{0,1-\epsilon} \lesssim \|\phi - \phi_{h,\Delta t}\|^2_{2,-\frac{1}{2}}$$

The upper bound is independent of the approximation method: TDBEM, convolution quadrature, no assumption on mesh.

The lower bound holds on quasi-uniform meshes.

BEM for waves: computable error estimates

Theorem

Let $\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that
$\mathcal{R} = \partial_t\ f - \mathcal{V}\partial_t \phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^1(\Gamma)) \Longrightarrow$

$$\|\phi - \phi_{h,\Delta t}\|^2_{0,-\frac{1}{2}} \lesssim \sum_{i,\Delta} \max\{\Delta t, h_\Delta\}\ \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

$$\max\{\Delta t, h\}\|\mathcal{R}\|^2_{0,1-\epsilon} \lesssim \|\phi - \phi_{h,\Delta t}\|^2_{2,-\frac{1}{2}}$$

Residual error indicators (RB):

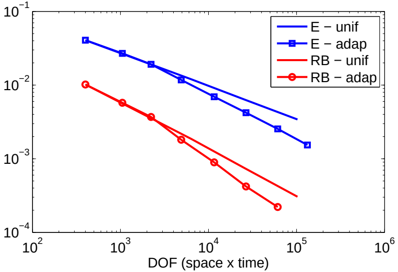
$$\eta^2(\Delta, i) = \max\{\Delta t, h_\Delta\}\ \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

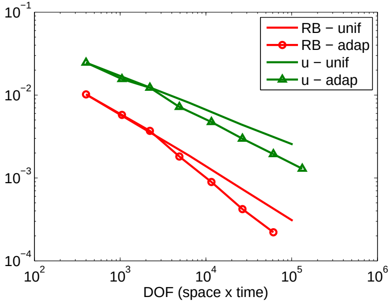
$$\eta^2 = \int_{t_i}^{t_{i+1}} \int_\Delta \Big\{ \left[\partial_t V\phi(t_i,\mathbf{x}) - \partial_t f(t_i,\mathbf{x})\right]^2 + \left[\nabla_T V\phi(t_i,\mathbf{x}) - \nabla_T f(t_i,\mathbf{x})\right]^2 \Big\}$$

Adaptivity: Wave Scattering off Triangular Cracks (1)

$\mathcal{V}\phi = \sin^5(t)$ on $\Gamma = 30 - 60 - 90$ Triangle, $0 < t < 2.5$.

Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.

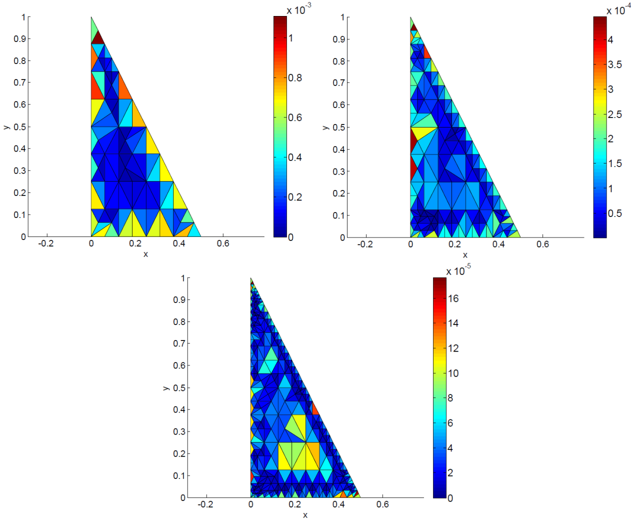




- Convergence rate $h^{0.45}$ (uniform), $h^{0.73}$ adaptive, as for time-independent problems.

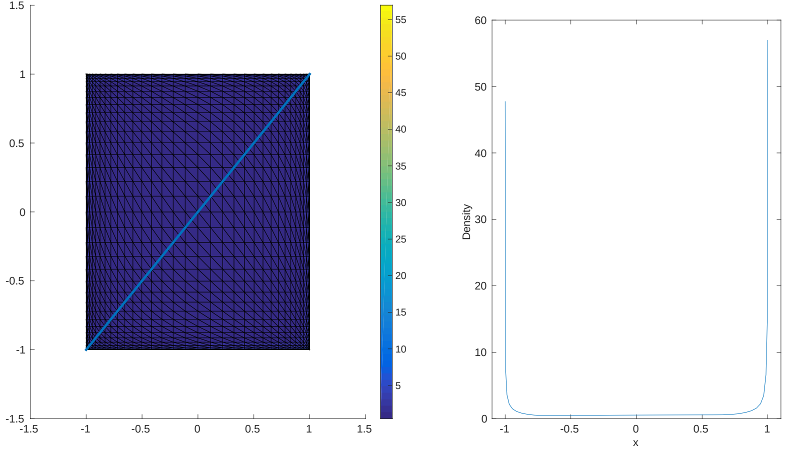
Adaptivity: Wave Scattering off Triangular Cracks

$\mathcal{V}\phi = \sin^5(t)$ on $\Gamma = 30 - 60 - 90$ Triangle, $0 < t < 2.5$.



Crack problems for waves: Edge and corner singularities

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$.
solution near corner $r^{-0.703\ldots}$, near edge $r^{-\frac{1}{2}}$



Crack problems for waves: edge and corner singularities

Time-harmonic waves: (Kondratiev, Dauge, ...)
Solution behaves like

- $r^{\gamma-1}$ near corner, γ=0.29 for square screen
- $r^{-\frac{1}{2}}$ near edge.

BEM on graded meshes $\Longrightarrow$ optimal approximation.

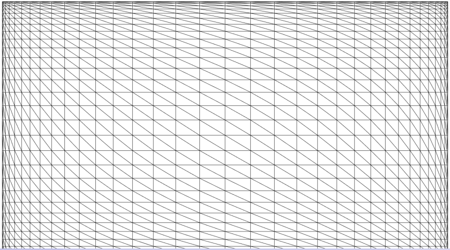
Crack problems for waves: edge and corner singularities

Theorem

a) Solution has same leading singular behaviour as in time-independent case.

b) Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) = \mathcal{O}(h^{\min\{\frac{\beta}{2},\frac{3}{2}\}-\varepsilon})$.

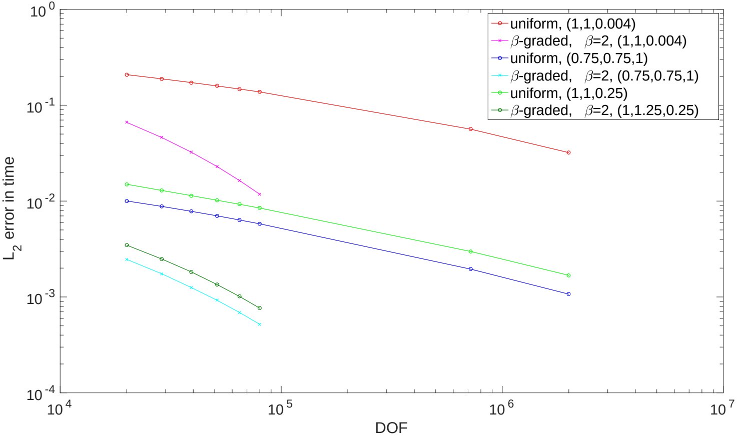
$$x_j = 1 - \left(\frac{j}{N}\right)^\beta,\ j = 1,\ldots,N\ .$$



Wave scattering off a crack: error of numerical solutions

Crack problem $\sim -(-\Delta)^{-1/2}u = f$
$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $f = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 0.5$.

error $\|u(x_0,t) - u_{h,\Delta t}(x_0,t)\|_{L^2_t}$: uniform vs. graded meshes



HG, Meyer, Özdemir, Stark, Stephan, Numerische Mathematik (2018).

Conclusions for cracks

Analysis & numerics: convergence rates for crack problems
(in degrees of freedom on a 2d crack, error measured in energy norm)

- 0.5: h-version, uniform
- 0.77: h-version, adaptive
- $\frac{\beta}{2}$: h-version, β-graded, $\beta \in [1, 3)$

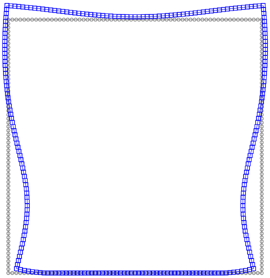
Coulomb friction: computable error estimate

Theorem

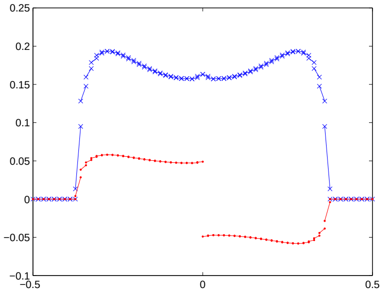
Let $\mathcal{F} \geq 0$ constant, $\lambda_t = \mathcal{F}\lambda_n\xi$, $\xi \in Dir_t(u_t)$, where $Dir_t(u_t)$ subdifferential of $u_t \mapsto |u_t|$, and assume $\mathcal{F}\,\|\xi\|$ is sufficiently small

$$
\begin{aligned}
&\|u-u^{hp}\|^2_{\tilde{H}^{1/2}(\Gamma_\Sigma)} + \|\psi-\psi^{hp}\|^2_{H^{-1/2}(\Gamma)} + \|\lambda^{kq}-\lambda\|^2_{\tilde{H}^{-1/2}(\Gamma_C)} \\
&\lesssim \sum_{E\in\mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E}\left\|f - S_{hp}u^{hp}\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E}\left\|\lambda^{kq} + S_{hp}u^{hp}\right\|^2_{L^2(E)} \\
&+ \sum_{E\in\mathcal{T}_{h,\Gamma}} h_E\left\|\frac{\partial}{\partial s}\left(V\psi^{hp} - (K+\tfrac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)} + \left\langle\left(\lambda_n^{kq}\right)^+, \left(g-u_n^{hp}\right)^+\right\rangle \\
&+ \left\|\left(g-u_n^{hp}\right)^-\right\|^2_{H^{1/2}(\Gamma_C)} + \left\|\left(\lambda_n^{kq}\right)^-\right\|^2_{\tilde{H}^{-1/2}(\Gamma_C)} \\
&+ \left\|\left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^+\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} - \left\langle\left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^-, \left|(u^{hp})_t\right|\right\rangle \\
&- \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} + \left\langle\left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C}.
\end{aligned}
$$

Coulomb friction: displacement and forces



(a) Reference (circle), deformed (square)



(b) λ_n (cross), λ_t (dot)

Figure: Solution of the Coulomb-friction problem, uniform mesh 256 elements, $p = 1$ (GLL/Bernstein)

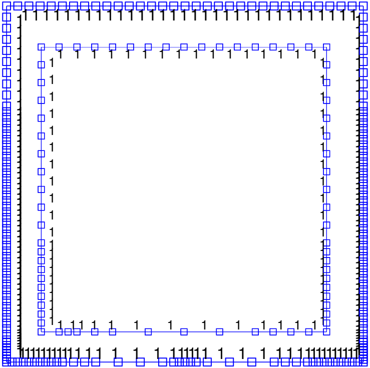
Coulomb friction: Error estimator vs. γ_0

$\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$, $\Gamma_C = [-\frac{1}{2}, \frac{1}{2}] \times \left\{-\frac{1}{2}\right\}$, $\Gamma_N = \partial\Omega \setminus \Gamma_C$
Elasticity parameters $E = 5$, $\nu = 0.45$, friction coefficient 0.3.

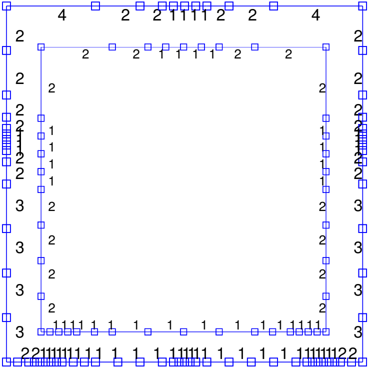
$$t_{\mathsf{side}} = \begin{pmatrix} -10\,\mathrm{sign}(x_1)(\frac{1}{2} + x_2)(\frac{1}{2} - x_2)\exp(-10(x_2 + \frac{4}{10})^2) \\ \frac{7}{8}(\frac{1}{2} + x_2)(\frac{1}{2} - x_2) \end{pmatrix}$$

$$t_{\mathsf{top}} = \begin{pmatrix} 0 \\ -\frac{25}{2}(\frac{1}{2} - x_1)^2(\frac{1}{2} + x_1)^2 \end{pmatrix}$$

Coulomb friction: meshes



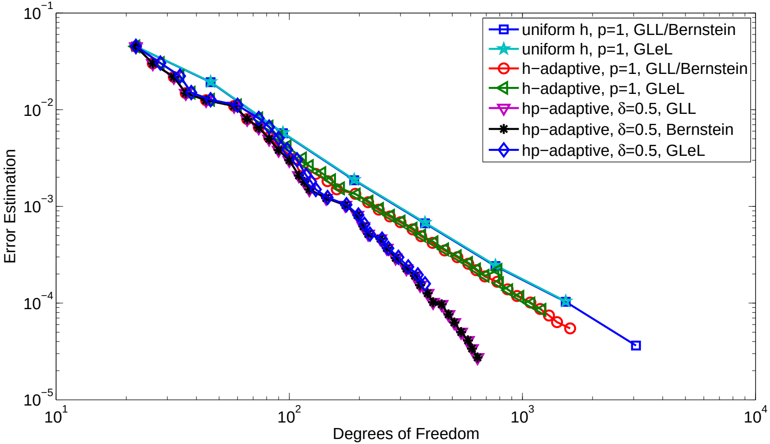
(a) h-adap. (GLL/Bernstein), mesh nr. 16 (in), 25 (out)



(b) hp-adap. (Bernstein), mesh nr. 16 (in), nr. 25 (out)

Figure: Adaptively generated meshes (Coulomb friction)

Coulomb friction: error of h- and hp-adaptive methods



Conclusions

- Nonlocal operators with contact and in singular domains
  choice of adapted meshes:
  graded meshes for simple cracks, adaptive for complex problems
- Example: Adaptivity and mesh refinements
  for nonlinear fractional contact problems
- Adaptivity: waves + elasticity
  (for today no coupling to local problems)

HG, J. Stocek, Space-time adaptive finite elements for nonlocal parabolic variational inequalities, in preparation.

HG, F. Meyer, C. Oezdemir, D. Stark, E. P. Stephan, Boundary elements with mesh refinements for the wave equation, Numer. Math. (2018).

L. Banz, HG, A. Issaoui, E. P. Stephan, Stabilized mixed hp-BEM for frictional contact problems in linear elasticity, Numer. Math. (2017).

See also http://www.macs.hw.ac.uk/∼hg94.