Solving approximate cloaking problems using finite element methods

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Abstract

Motivating from the approximate cloaking problem, we consider a variable coefficient Helmholtz equation with a fixed wave number. We use finite element methods to discretize the equation. Numerical results show the numerical solutions exhibit the cloaking behaviours.

Contents

1 Introduction

Since 2005 there has been a wave of serious theoretical proposals [1, 6] in the physics literature, and a widely reported experiment by Schurig et al. [9], for cloaking devices – structures that would not only render an object invisible but also undetectable to electromagnetic waves. The mathematical foundations of optical cloaking are described in an excellent article by Greenleaf et.al. [3].

The transformation optics approach to cloaking uses a singular change of coordinates [4, 8], which blows up a point to the region being cloaked, is singular and hence it is difficult to analyse theoretically. Hence a rigorous numerical simulation will shed the light on the problem significantly.

In this paper, we will briefly review the theoretical background of the approximate cloaking problem and propose a finite element method to solve the problem numerically. While there were some papers e.g. [2] describing cloaking experiments using the commericial finite-element COMSOL Multiphysics, proper mathematical explanations were not available there. Here, we offer a summary on the theory as well a finite element method using an open source package.

2 Mathematical problem

Let Ω be a bounded domain in $\mathbb{R}^n$ for $n = 2, 3$. Light waves go through the domain Ω can be described by the wave equation

$$q(x)U_{tt} - \nabla \cdot (A(x)\nabla U) = 0.$$

With the harmonic solutions $U = ue^{-ikt}$ we obtain the scalar Helmholtz equation

$$\nabla \cdot (A(x)\nabla u) + k^2 q(x)u = 0 \text{ in } \Omega. \tag{1}$$

The solution to the Helmholtz equation (1) is uniquely defined if either the Dirichlet condition

$$u = g \text{ on } \partial\Omega, \tag{2}$$

or the Neumann condition

$$\frac{\partial u}{\partial n} = \psi \text{ on } \partial\Omega \tag{3}$$

is given.

With respect to the Helmholtz equation (1), we define the map $\Lambda_{A,q} : H^{-1/2}(\Omega) \to H^{1/2}(\Omega)$ by

$$\begin{aligned} &\Lambda_{A,q}(\psi) = u|_{\partial\Omega} \\ &u \text{ solves (1) with } \sum A_{ij}\frac{\partial u}{\partial x_j}\nu_i = \psi \text{ on } \partial\Omega \end{aligned} \tag{4}$$

Let B_r be the open ball of radius r, that is, $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. Suppose the given domain Ω containing B_2. A specific structure $A_c(x), q_c(x)$

defined on the shell $B_2 \setminus B_1$ is said to cloak the unit ball B_1 if whenever

$$A(x), q(x) = \begin{cases} I, 1 & \text{for } x \in \Omega \setminus B_2, \\ A_c, q_c & \text{in } B_2 \setminus B_1, \\ \text{arbitrary} & \text{in } B_1, \end{cases} \tag{5}$$

then

$$\Lambda_{A,q} = \Lambda_{I,1}.$$

In other words, the boundary measurements (Dirichlet and Neumann data) on $\partial\Omega$ with respect to $A(x), q(x)$ are identical to those obtained when $A = I, q = 1$. Physically, Ω looks uniformly regarless the content of B_1. Or we can say that light waves at the boundary $\partial\Omega$ behaving the same way regarless the content of B_1, giving the impression that B_1 is cloaked.

In [9], a change of variable scheme is proposed to construct a cloak A_c, q_c. The scheme relies on the following fact [4]:

Let $F : \Omega \to \Omega$ be a differentiable, orientation-preserving, surjective and invertible map such that $F(x) = x$ on $\partial\Omega$. Let

$$F_*A(y) = \frac{DF(x)A(x)DF^T(x)}{\det DF(x)}, \quad F_*q(y) = \frac{q(x)}{\det DF(x)}, \quad x = F^{-1}(y)$$

Then

$$u(x) \text{ solves } \nabla_x \cdot (A(x)\nabla_x u) + k^2 q(x) u = 0$$

if and only if

$$w(y) = u(F^{-1}(y)) \text{ solves } \nabla_y \cdot (F_*A(y)\nabla_y w) + k^2 F_*q(y) w = 0.$$

Moreover A, q and F_*A, F_*q give the same boundary measurements,

$$\Lambda_{A,q} = \Lambda_{F_*A, F_*q}.$$

An example of the map F is given by $F = F_\varepsilon$, (see [9]), where

$$F_\varepsilon(x) = \begin{cases} \frac{x}{\varepsilon} & \text{if } |x| \le \varepsilon \\ \left(\frac{2-2\varepsilon}{2-\varepsilon} + \frac{|x|}{2-\varepsilon}\right) \frac{x}{|x|} & \text{if } \varepsilon \le |x| \le 2 \\ x & \text{if } x \in \Omega \setminus B_2. \end{cases} \tag{6}$$

We can see that F_ε maps B_ε to the unit ball B_1, the annulus $B_2 \setminus B_\varepsilon$ to the annulus $B_2 \setminus B_1$ and outside B_2 the map F_ε is just the identity map.

The inverse map F_ε^{-1} is given by

$$F_\varepsilon^{-1}(y) = \begin{cases} \varepsilon y & \text{if } |y| \le 1 \\ y\left(2 - \varepsilon - \frac{2(1-\varepsilon)}{|y|}\right) & \text{if } 1 \le |y| \le 2 \end{cases} \tag{7}$$

It is suggested that (as in [9, 3]) if we take $F_0 = \lim_{\varepsilon\to 0} F_\epsilon$, i.e. F_0 is the singular map that blows up to origin to the ball B_1, and define

$$A_c = (F_0)_* I, \quad q_c = (F_0)_* 1,$$

then the ball B_1 would be cloaked. So it is logically to think that for small ε, then $(F_\varepsilon)_* I$, $(F_\varepsilon)_* 1$ should nearly cloak B_1, which means that if

$$A(y), q(y) = \begin{cases} I,\ 1 & \text{for } y \in \Omega \setminus B_2, \\ (F_\varepsilon)_* I, (F_\varepsilon)_* 1 & \text{for } y \in B_2 \setminus B_1, \\ \text{arbitrary} & \text{for } y \in B_1, \end{cases} \tag{8}$$

then $\Lambda_{A,q} \approx \Lambda_{1,1}$. However, the statement is not true for $k \neq 0$ (see [4, Section 2.5]) due to resonance.

To explain this point further, let $\Omega = B_2$ and consider

$$A_\varepsilon, q_\varepsilon = \begin{cases} I, 1 & \text{in } B_2 \setminus B_\varepsilon, \\ \tilde{A}_\varepsilon, \tilde{q}_\varepsilon & \text{in } B_\varepsilon, \end{cases}$$

where $\tilde{A}_\varepsilon$ and $\tilde{q}_\varepsilon$ are real-valued constants. The general solution of the associated 2D Helmholtz equation can be expressed in polar coordinates as

$$u = \begin{cases} \sum_{\ell=-\infty}^{\infty} \alpha_\ell J_\ell\left(kr\sqrt{\tilde{q}_\varepsilon/\tilde{A}_\varepsilon}\right) e^{i\ell\theta} & \text{for } r \le \varepsilon \\ \sum_{\ell=-\infty}^{\infty} [\beta_\ell J_\ell(kr) + \gamma_\ell H_\ell^{(1)}(kr)] e^{i\ell\theta} & \text{for } \varepsilon < r \le 1, \end{cases}$$

for appropriate choices of α_ℓ, β_ℓ, and γ_ℓ. Here J_ℓ and $H_\ell^{(1)}$ are the classical Bessel and Hankel functions of the first kind, respectively. When we solve a Neumann problem, the three unknowns at mode ℓ $(\alpha_\ell, \beta_\ell, \gamma_\ell)$ are determined by three linear equations: agreement with the Neumann data at $r = 2$ and satisfaction of the two transmission conditions at $r = \varepsilon$. However, for any $k \neq 0$ and any ℓ, this linear system has zero determinant at selected values of $\tilde{A}_\varepsilon, \tilde{q}_\varepsilon$. When the linear system is degenerate (for some ℓ) the homogeneous Neumann problem has a nonzero solution, and the boundary map $\Lambda_{A_\varepsilon, q_\varepsilon}$ is not even well-defined. In other words, no matter how small the value of ε, for any $k \neq 0$ there are cloak-busting choices of $\tilde{A}_\varepsilon, \tilde{q}_\varepsilon$ for which the ball with such an inclusion is resonant at frequency k.

To deal with the resonance problem, a near-cloak mechanism is introduced in [4], which has a new damping parameter $\beta > 0$. The near-cloak is defined as follows

$$A(y), q(y) = \begin{cases} I, 1 & \text{for } y \in \Omega \setminus B_2, \\ (F_{2\varepsilon})_* I, (F_{2\varepsilon})_* 1 & \text{for } y \in B_2 \setminus B_1, \\ (F_{2\varepsilon})_* I, (F_{2\varepsilon})_* (1 + i\beta) & \text{for } y \in B_1 \setminus B_{1/2}, \\ \text{arbitrary real, elliptic} & \text{for } y \in B_{1/2}. \end{cases} \tag{9}$$

With $\beta > 0$, the following problem is well-posed ([4, Proposition 3.5])

$$\begin{cases} \nabla(A_\varepsilon \nabla u) + k^2 q_\varepsilon u = 0 & \text{in } \Omega \\ \partial u / \partial \nu = \psi & \text{in } \partial\Omega, \end{cases} \tag{10}$$

where

$$\begin{cases} A_\varepsilon = I, \ q_\varepsilon = 1 & \text{in } \Omega \setminus B_{2\varepsilon}, \\ A_\varepsilon = 1, \ q_\varepsilon = 1 + i\beta & \text{in } B_{2\varepsilon} \setminus B_\varepsilon, \\ A_\varepsilon, q_\varepsilon \text{ arbitrary real, elliptic} & \text{in } B_\varepsilon. \end{cases}$$

Furthermore, when $\beta \sim \varepsilon^{-2}$, then their construction approximately cloaks $B_{1/2}$ in the sense that

$$\|\Lambda_{A,q} - \Lambda_{I,1}\| \leq C \begin{cases} 1/|\log \varepsilon| & \text{in space dimension 2,} \\ \varepsilon & \text{in space dimension 3.} \end{cases} \tag{11}$$

The theoretical estimate (11) is pessimistic in two dimension since the proof relies on the fundamental solution of the 2D Laplace equation. However, numerical experiments show that when $\varepsilon \to 0$, the approximate cloaking scheme performs reasonably well.

3 Using finite element methods

In this section, we will describe how to solve the near cloaking problem using finite element methods. The weak formulation of (10) is: find $u \in H^1(\Omega)$ so that

$$\int_\Omega [A_\varepsilon(x) \nabla_x u(x) \cdot \nabla_x v(x) - k^2 q_\varepsilon u(x) v(x)] dx = \int_{\partial\Omega} A_\varepsilon \psi v dx, \quad \forall v \in H^1(\Omega). \tag{12}$$

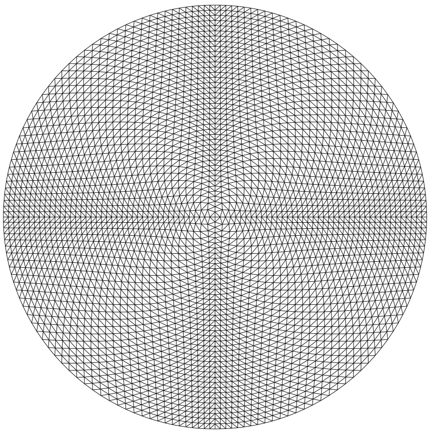

Figure 1: A uniform mesh for computing U_ε on B_2

The weak formulation of the push forward problem is: find $w \in H^1(\Omega)$ so that

$$\int_\Omega [F_*(A_\varepsilon)\nabla_y w(y)\cdot\nabla_y\phi(y) - k^2 q_\varepsilon w(y)\phi(y)]dy = \int_{\partial\Omega} F_* A_\varepsilon \psi\phi dx, \quad \forall\phi \in H^1(\Omega). \tag{13}$$

Introducing the bilinear form

$$a(w,\phi) = \int_\Omega [F_*(A_\varepsilon)\nabla_y w(y)\cdot\nabla_y\phi(y) - k^2 q_\varepsilon w(y)\phi(y)]dy$$

and defining the finite dimensional space

$$V_h = \mathrm{span}\{\phi_1, \phi_2, \ldots, \phi_N\} \subset H^1(\Omega),$$

the Ritz-Galerkin approximation problem to the push forward problem (13) is written as: find $w \in V_h$ so that

$$a(w,\chi) = \int_{\partial\Omega} A_\varepsilon\psi\chi \quad \forall\chi \in V_h(\Omega).$$

A uniform mesh that is used to constructed the piecewise linear finite elements when $\Omega = B_2$ is shown in Figure 1.

4 Numerical experiments

In this section, we describe some initial numerical experiments of the following interior 2-D Dirichlet problem, following an example in [4].

$$\begin{aligned} \nabla\cdot(F_*(A_\varepsilon)\nabla U_\varepsilon(y)) + k^2 F_*(q_\varepsilon)U_\varepsilon(y) &= 0 \qquad \text{in } B_2, \\ \partial_\nu U_\varepsilon(y) &= \partial_\nu u_0(2,\theta) \quad \text{on } \Gamma = \partial B_2. \end{aligned} \tag{14}$$

where

$$u_0(r,\theta) = \sum_{\ell=-30}^{30} J_\ell(kr)e^{i\ell\theta},$$

where J_ℓ is the classical Bessel function of order ℓ.

In 2D,

$$\begin{cases} F_*(A_\varepsilon)(y) = \frac{1}{\det DF(x)} DF(x)DF^T(x)|_{x=F^{-1}(y)} & \\ F_*(q_\varepsilon)(y) = \frac{1}{\det DF(x)}|_{x=F^{-1}(y)}, & \text{for } 1 < |y| \le 2 \\ F_*(A_\varepsilon)(y) = 1, \qquad F_*(q_\varepsilon)(y) = 4\varepsilon^2(1+i\beta), & \text{for } \frac{1}{2} < |y| \le 1 \\ F_*(A_\varepsilon(y)) = A_\varepsilon & \\ F_*(q_\varepsilon(y)) = 4\varepsilon^2 q_\varepsilon & \text{for } |y| \le \frac{1}{2}. \end{cases}$$

We now compute the Jacobian $F' = DF(x) = (\partial F_i/\partial x_j)$. The computation for the special case $\varepsilon = 0$ can be found in [5].

$$DF(x) = \left[\left(\frac{1-2\varepsilon}{1-\varepsilon}\right)\frac{1}{|x|} + \frac{1}{2(1-\varepsilon)}\right] I - \left(\frac{1-2\varepsilon}{1-\varepsilon}\right)\frac{\hat{x}\hat{x}^T}{|x|}, \tag{15}$$

where $\hat{x} = x/|x|$ and I is the identity matrix.

To find the determinant for $DF(x)$, we note that $\hat{x}$ is an eigenvector of $DF(x)$ with eigenvalue $0.5/(1-\varepsilon)$ and $\hat{x}^\perp$ is an $n-1$ dimensional eigenspace with eigenvalue

$$\frac{(1-2\varepsilon)}{(1-\varepsilon)}\frac{1}{|x|} + \frac{1}{2(1-\varepsilon)}$$

So, the determinant of $DF(x)$ is

$$\det\, DF(x) = \frac{1}{2(1-\varepsilon)}\left[\frac{(1-2\varepsilon)}{(1-\varepsilon)}\frac{1}{|x|} + \frac{1}{2(1-\varepsilon)}\right]^{n-1}$$

At $x = F^{-1}(y)$, we have

$$\frac{1}{\det\, DF(F^{-1}(y))} = \frac{2}{|y|^2}\big(2|y|(1-\varepsilon) - 2(1-2\varepsilon)\big)^2$$

The following product $DF(x)(DF(x))^T$

$$\begin{aligned} DF(x)(DF(x))^T = &\left(\frac{(1-2\varepsilon)^2}{(1-\varepsilon)^2}\frac{1}{|x|^2} + \frac{(1-2\varepsilon)}{(1-\varepsilon)^2}\frac{1}{|x|} + \frac{1}{4(1-\varepsilon)^2}\right) I \\ &- \left(\frac{(1-2\varepsilon)^2}{(1-\varepsilon)^2}\frac{1}{|x|^2} + \frac{(1-2\varepsilon)}{(1-\varepsilon)^2}\frac{1}{|x|}\right)\hat{x}\hat{x}^T \end{aligned}$$

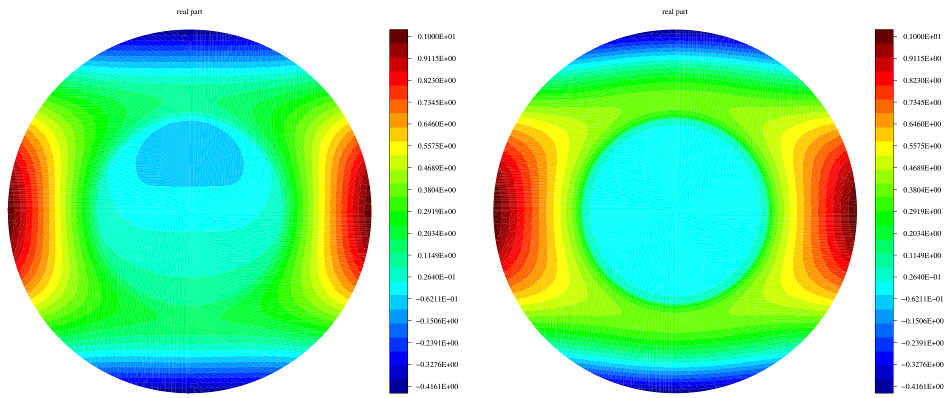

Figure 2: The 2D push-forward FEM solutions U_ε ($\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-6}$) on B_2

For the Dirichlet problem we have performed numerical simulations with the program package MAIPROGS [7] using FEM-2D with piecewise linear elements. We set the wave number $k = 1$ and $A_\varepsilon = q_\varepsilon = 1$.

In Figure 2 the finite element approximation u_h of U_ε with $\varepsilon = 10^{-d}$, $d = 1, 6$ and $\beta = \varepsilon^{-2}$ are shown. As ε get smaller, the numerical solutions stay away from $B_{1/2}$, so the content of $B_{1/2}$ is cloaked. They are consistent with numerical results using a different method published in [4, Section 4].

Conclusions In this work, we have summarized the approximate cloaking framework proposed in [4] and proposed a finite element method to construct numerical solutions using an open source package. It will lay a foundation for future work in error analysis or coupled finite element method/ boundary element methods for the problem.

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