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The skeleton equation method for acoustic transmission problems with varying coefficients

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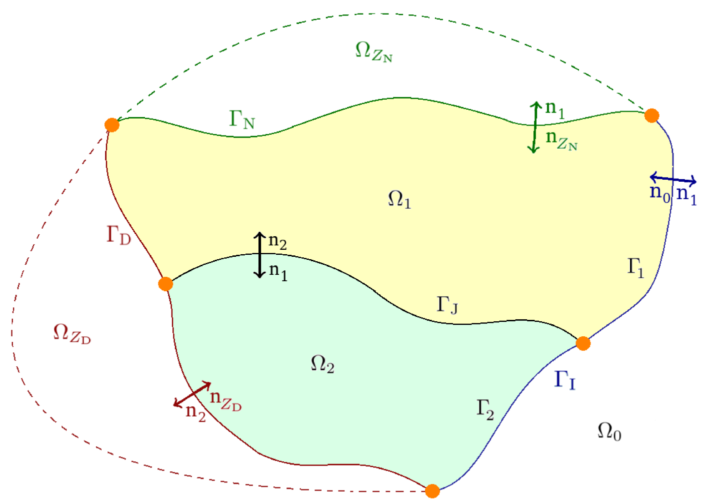
August 2022

joint work with F. Florian (U Zürich), R. Hiptmair (ETH Zürich)

Acoustic wave transmission problems in frequency domain

Geometric setting:



$\Omega \subset \mathbb{R}^3$ bounded Lipschitz domain with boundary Γ,
$\Omega_1 \cup \Gamma_J \cup \Omega_2$ subdomains and jump interface,
$\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_I$: Dirichlet-, Neumann, impedance part of the boundary,
$\Gamma_1 := \partial\Omega_1$ and $\Gamma_2 := \partial\Omega_2$.

PDE:

$$-\,\mathsf{div}\left(\mathbb{A}_j \nabla u_j\right) + p_j^2 s^2 u_j = 0 \quad \text{in } \Omega_j, \quad j = 1, 2.$$

$$[\gamma_{\mathsf{D}} u]_{\Gamma_{\mathsf{J}}} = [\gamma_{\mathbb{A}}]_{\Gamma_{\mathsf{J}}} = 0 \quad \text{on } \Gamma_{\mathsf{J}},$$

$$\gamma_{\mathsf{D}} u = g_{\mathsf{D}} \quad \text{on } \Gamma_{\mathsf{D}},$$

$$\gamma_{\mathbb{A}} u = d_{\mathsf{N}} \quad \text{on } \Gamma_{\mathsf{N}},$$

$$\gamma_{\mathbb{A}} u + sTu = d_{\mathsf{I}} \quad \text{on } \Gamma_{\mathsf{I}},$$

Assumption on the coefficients

Assumption. For $j = 1, 2$, the coefficients in the PDE satisfy

1. $\mathbb{A}_j \in \mathbb{L}^{\infty}_{>0}\left(\Omega_j, \mathbb{R}^{3\times 3}_{\mathsf{sym}}\right)$,
2. $p_j \in L^{\infty}_{>0}\left(\Omega_j\right)$,
3. $s \in \mathbb{C}_{>0} := \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta > 0\}$ and $|s| \geq s_0$ for some $s_0 > 0$.

Goal:

Introduce a **transform** of the PDE to an integral equation on the skeleton $\Sigma := \partial\Omega_1 \cup \partial\Omega_2$ such that

a) the resulting skeleton equation is coercive and elliptic,

b) the explicit knowledge of a fundamental solution/Green's function is not required.

Remark: There are many approaches to transform boundary value problems to integral equations:
a) direct/indirect method
b) first kind/second kind integral equation
c) scalar equation/symmetric coupling/non-symmetric coupling

and the proof of well-posedness can be a subtle issue.

Abstract layer potentials:

Sesquilinear forms:

$$\ell : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C} \qquad \ell(s)(u,v) := \left\langle \mathbb{A}\nabla u, \overline{\nabla v} \right\rangle_{\mathbb{R}^3} + \left\langle p_j s^2 u, \overline{v} \right\rangle_{\mathbb{R}^3}.$$

$$\ell_j : H^1(\Omega_j) \times H^1(\Omega_j) \to \mathbb{C} \qquad \ell_j(s)(u,v) := \left\langle \mathbb{A}_j\nabla u, \overline{\nabla v} \right\rangle_{\Omega_j} + \left\langle p_j s^2 u, \overline{v} \right\rangle_{\Omega_j}.$$

Coercivity and continuity of $\ell(\cdot,\cdot)$, $\ell_j(\cdot,\cdot)$

Lemma (Bamberger/Ha-Duong 1986), . For $j = 1, 2$ and $\mu := s/|s|$:

$$\left|\ell(s)(v,w)\right| \leq \Lambda \left\|\!\left|v\right|\!\right\|_{\mathbb{R}^3;s} \left\|\!\left|w\right|\!\right\|_{\mathbb{R}^3;s} \qquad \operatorname{Re}\ell_j(s)(v,\mu v) \geq \lambda \frac{\operatorname{Re}s}{|s|} \left\|\!\left|v\right|\!\right\|^2_{\mathbb{R}^3;s},$$

$$\left|\ell_j(s)\left(v_j,w_j\right)\right| \leq \Lambda \left\|\!\left|v_j\right|\!\right\|_{\Omega_j;s} \left\|\!\left|w_j\right|\!\right\|_{\Omega_j;s} \qquad \operatorname{Re}\ell_j(s)\left(v_j,\mu v_j\right) \geq \lambda \frac{\operatorname{Re}s}{|s|} \left\|\!\left|v_j\right|\!\right\|^2_{\Omega_j;s},$$

with

$$\lambda := \min\left\{\lambda(p), \lambda(\mathbb{A})\right\} \quad \text{and} \quad \Lambda := \max\left\{\Lambda(p), \Lambda(\mathbb{A})\right\}.$$

and

$$\left\|\!\left|w\right|\!\right\|_{\Omega;s} := \left(\|\nabla w\|^2_{L^2(\Omega)} + |s|^2 \|w\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Associated operators:

$$\mathsf{L} : H^1\left(\mathbb{R}^3\right) \to H^{-1}\left(\mathbb{R}^3\right) \qquad (\mathsf{L}u)(v) := \ell(u,v),$$

$$\mathsf{L}_j : H^1\left(\mathbb{R}^3\right) \to H^{-1}\left(\mathbb{R}^3\right), \qquad \left(\mathsf{L}_j u\right)(v) := \ell_j\left(u|_{\Omega_j}, v|_{\Omega_j}\right),$$

$$H^1_{j;0}\left(\mathbb{R}^3\right) := \left\{w \in H^1\left(\mathbb{R}^3\right) : w|_{\Gamma_j} = 0\right\} \quad H^{-1}_j\left(\mathbb{R}^3\right) := \left(H^1_{j;0}\left(\mathbb{R}^3\right)\right)'.$$

$$\mathsf{L}_{\mathsf{D};j}(s) : H^1\left(\Omega_j\right) \to H^{-1}_j\left(\mathbb{R}^3\right) \qquad \left\langle \mathsf{L}_{\mathsf{D};j}(s)\, v, \overline{w}\right\rangle_{\mathbb{R}^3} = \ell_j(s)\left(v, w|_{\Omega_j}\right) \quad \forall w \in H^1_{j;0}\left(\mathbb{R}^3\right).$$

Corollary. The operator $\mathsf{L}_{\mathsf{D};j}(s)$ applied to functions $v \in H^1(\Omega_j, \mathbb{A}_j) \subset H^1(\Omega_j)$ is the piecewise application of the differential operator in Ω_j :

$$\mathsf{L}_{\mathsf{D};j}(s)\, v := \begin{cases} -\operatorname{div}\left(\mathbb{A}_j \nabla v\right) + p_j s^2 v & \text{in } \Omega_j, \\ 0 & \text{in } \mathbb{R}^3 \backslash \overline{\Omega_j}. \end{cases}$$

Remark. Since the PDE has zero right-hand side, the solution satisfies $u_j \in H^1(\Omega_j, \mathbb{A}_j)$. The operator form of this equation is given by

$$\mathsf{L}_{\mathsf{D};j}(s)\, u_j = 0 \qquad \text{in } \Omega_j, \quad j = 1, 2.$$

Layer potentials:

a) Single layer potential

We employ the approach by *A. Barton, Elect. J. Diff. Eq., 2017*, for our setting.

Definition. The solution operator (acoustic Newton potential) $\mathsf{N}(s) : H^{-1}\left(\mathbb{R}^3\right) \rightarrow H^1\left(\mathbb{R}^3\right)$ is given by the relation

$$\ell(s)\left(\mathsf{N}(s) f, w\right) = \langle f, \overline{w} \rangle_{\mathbb{R}^3} \qquad \forall f \in H^{-1}\left(\mathbb{R}^3\right), \quad \forall w \in H^1\left(\mathbb{R}^3\right).$$

Lemma. The Newton potential is a left inverse of $\mathsf{L}(s)$, i.e.,

$$v = \mathsf{N}(s) \circ \mathsf{L}(s) v = \mathsf{N}(s) \circ \mathsf{L}_1(s) v + \mathsf{N}(s) \circ \mathsf{L}_2(s) v \quad \forall v \in H^1\left(\mathbb{R}^3\right)$$

and satisfies the estimate

$$|||\mathsf{N}(s) f|||_{\mathbb{R}^3;s} \leq \frac{|s|}{\lambda \operatorname{Re} s} \|f\|_{H^{-1}(\mathbb{R}^3);s} \quad \forall f \in H^{-1}\left(\mathbb{R}^3\right).$$

Definition. For $j = 1, 2$ and $\varphi \in H^{-1/2}\left(\Gamma_j\right)$ the *single layer potential* $\mathsf{S}_j(s) : H^{-1/2}\left(\Gamma_j\right) \rightarrow H^1\left(\mathbb{R}^3\right)$ is given by

$$\mathsf{S}_j(s)\,\varphi := \mathsf{N}(s)\left(\gamma_{\mathsf{D};j}(s)\right)'\varphi.$$

Alternative definition:

Lemma. For $\varphi \in H^{-1/2}\left(\Gamma_j\right)$, it holds $\mathsf{S}_j(s)\,\varphi \in H^1\left(\mathbb{R}^3\right)$ and

$$\ell(s)\left(\mathsf{S}_j(s)\,\varphi, v\right) = \left\langle \varphi, \gamma_{\mathsf{D};j}(s)\,\overline{v}\right\rangle_{\Gamma_j} \quad \forall v \in H^1\left(\mathbb{R}^3\right).$$

b) Double layer potential.

Definition. Let $\varphi \in H^{1/2}(\Gamma)$ and $f \in H^1\left(\mathbb{R}^3\right)$ such that $\gamma_{\mathsf{D};j}(s) f = \phi$. Then, the *double layer potential* $\mathsf{D}_j(s) : H^{1/2}\left(\Gamma_j\right) \to H^1\left(\Omega_j\right) \times H^1\left(\Omega_{j'}\right)$ (with $j' := 3 - j$) is given by

$$\mathsf{D}_j(s)\,\varphi\Big|_{\Omega_j} := -\left.f\right|_{\Omega_j} + \left(\mathsf{N}(s)\,\mathsf{L}_j(s)\,f\right)\Big|_{\Omega_j},$$

$$\mathsf{D}_j(s)\,\varphi\Big|_{\Omega_{j'}} := \left.f\right|_{\Omega_{j'}} - \left(\mathsf{N}(s)\,\mathsf{L}_j(s)\,f\right)\Big|_{\Omega_{j'}}.$$

These abstract potentials satisfy the homogeneous PDE:

Lemma. For any $\varphi \in H^{-1/2}\left(\Gamma_j\right)$, $\psi \in H^{1/2}\left(\Gamma_j\right)$ it holds for $j, m \in \{1, 2\}$

$$\mathsf{L}_{\mathsf{D};j}(s)\,\mathsf{S}_m(s)\,\varphi = \mathsf{L}_{\mathsf{D};j}(s)\,\mathsf{D}_m(s)\,\psi = 0.$$

Lemma (Green's representation formula). Let $u \in H^1\left(\Omega_j, \mathbb{A}_j\right)$ and $\mathsf{L}_{\mathsf{D};j}(s)\,u = 0$. Then, the *Green's representation formulae* hold

$$u = \left(\mathsf{S}_j(s)\,\gamma_{\mathbb{A};j}(s)\,u - \mathsf{D}_j(s)\,\gamma_{\mathsf{D};j}(s)\,u\right)\Big|_{\Omega_j},$$

$$0 = \left(\mathsf{S}_j(s)\,\gamma_{\mathbb{A};j'}(s)\,u - \mathsf{D}_j(s)\,\gamma_{\mathsf{D};j}(s)\,u\right)\Big|_{\Omega_{j'}}.$$

Lemma. For any $\varphi \in H^{-1/2}\left(\Gamma_j\right)$ and $\psi \in H^{1/2}\left(\Gamma_j\right)$ the jump relations hold:

$$\left[\left(\mathsf{S}_j(s)\,\varphi\right)\right]_{\mathsf{D};j}(s) = 0, \qquad \left[\left(\mathsf{S}_j(s)\,\varphi\right)\right]_{\mathbb{A};j}(s) = -\varphi,$$

$$\left[\left(\mathsf{D}_j(s)\,\psi\right)\right]_{\mathsf{D};j}(s) = \psi, \qquad \left[\left(\mathsf{D}_j(s)\,\psi\right)\right]_{\mathbb{A};j}(s) = 0.$$

Calderón operators:

The application of the Cauchy trace to Green's representation formula results in the Calderón identity on the domain skeleton.

Definition. For $j = 1, 2$, the *skeleton operators* are given by

$$\mathsf{V}_j(s) : H^{-1/2}(\Gamma_j) \to H^{1/2}(\Gamma_j) \qquad \mathsf{V}_j(s)\varphi := \{\!\{\mathsf{S}_j(s)\varphi\}\!\}_{\mathsf{D};j}(s),$$

$$\mathsf{K}_j(s) : H^{1/2}(\Gamma_j) \to H^{1/2}(\Gamma_j) \qquad \mathsf{K}_j(s)\psi := \{\!\{\mathsf{D}_j(s)\psi\}\!\}_{\mathsf{D};j}(s),$$

$$\mathsf{K}'_j(s) : H^{-1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j) \qquad \mathsf{K}'_j(s)\varphi := \{\!\{\mathsf{S}_j(s)\varphi\}\!\}_{\mathbb{A};j}(s),$$

$$\mathsf{W}_j(s) : H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j) \qquad \mathsf{W}_j(s)\psi := -\{\!\{\mathsf{D}_j(s)\psi\}\!\}_{\mathbb{A};j}(s),$$

Cauchy traces and multi-trace space

$$\mathbf{X}_j := H^{1/2}(\Gamma_j) \times H^{-1/2}(\Gamma_j) \quad \text{for } j = 1, 2$$

$$\mathbf{X}^{\text{mult}} := \mathbf{X}_1 \times \mathbf{X}_2 \quad \text{multi trace space}$$

Remark: The mulit-trace space is multivariate on the interfaces.

Definition. The *Calderón operator* $\mathsf{C}(s) : \mathbf{X}^{\mathsf{mult}} \to \mathbf{X}^{\mathsf{mult}}$ is by

$$\mathsf{C}(s) := \operatorname{diag}\left[\mathsf{C}_j(s) : j = 1, 2\right] \quad \text{with} \quad \mathsf{C}_j(s) := \begin{bmatrix} -\mathsf{K}_j(s) & \mathsf{V}_j(s) \\ \mathsf{W}_j(s) & \mathsf{K}_j'(s) \end{bmatrix}.$$

The sesquilinear form $c(s) : \mathbf{X}^{\mathsf{mult}} \times \mathbf{X}^{\mathsf{mult}} \to \mathbb{C}$ associated to the operator $\mathsf{C}(s) - \frac{1}{2}\,\mathsf{Id}$ is

$$c(s)(\phi, \psi) := \sum_{j=1}^{2} \left\langle \begin{pmatrix} -\frac{1}{2}\phi_{\mathsf{D}} - \mathsf{K}_j(s)\,\phi_{\mathsf{D}} & +\mathsf{V}_j(s)\,\phi_{\mathsf{N}} \\ \mathsf{W}_j(s)\,\phi_{\mathsf{D}} & -\frac{1}{2}\phi_{\mathsf{N}} + \mathsf{K}_j'(s)\,\phi_{\mathsf{N}} \end{pmatrix}, \begin{pmatrix} \overline{\psi_{\mathsf{N}}}, \\ \overline{\psi_{\mathsf{D}}} \end{pmatrix} \right\rangle_{\Gamma_j}.$$

Multi trace and single trace formulation of the transmission problem:

Multi-trace formulation of original transmission problem (see Claeys et al., '15):

Find:

$$\mathbf{u}^{\mathsf{mult}} = \left(\mathbf{u}_j^{\mathsf{mult}}\right)_{j=1}^{2} = \left(\left(u_{\mathsf{D};j}^{\mathsf{mult}}, u_{\mathsf{N};j}^{\mathsf{mult}}\right)\right)_{j=1}^{2} \in \mathbf{X}^{\mathsf{mult}}$$

such that:

$$\left(\mathsf{C}_j\left(s\right) - \tfrac{1}{2}\,\mathsf{Id}_j\right)\mathbf{u}_j^{\mathsf{mult}} = 0 \qquad \text{in } \Omega_j \quad j = 1, 2,$$

$$\left[\mathbf{u}^{\mathsf{mult}}\right]_{1,2} = [\beta]_{1,2}$$

$$\left.\begin{array}{l} \left.u_{\mathsf{D};j}^{\mathsf{mult}}\right|_{\Gamma_j \cap \Gamma_{\mathsf{D}}} = \left.\beta_{\mathsf{D};j}\right|_{\Gamma_j \cap \Gamma_{\mathsf{D}}} \\ \left.u_{\mathsf{N};j}^{\mathsf{mult}}\right|_{\Gamma_j \cap \Gamma_{\mathsf{N}}} = \left.\beta_{\mathsf{N};j}\right|_{\Gamma_j \cap \Gamma_{\mathsf{N}}} \end{array}\right\} \quad j = 1, 2.$$

Final step, the single-trace formulation:

A single trace formulation is obtained if the transmission conditions are incorporated into the multi trace space $\mathbf{X}^{\text{mult}}$.

Advantages:

1) The sesquilinear form $c(s)(\cdot,\cdot)$ is coercive on $\mathbf{X}_0^{\text{single}} \times \mathbf{X}_0^{\text{single}}$ (but not on $\mathbf{X}^{\text{mult}} \times \mathbf{X}^{\text{mult}}$).

2) The functions on the interfaces become *single-valued.*

Definition (single trace space)

$$\mathbf{X}^{\mathsf{single}} := \left\{ \psi \in \mathbf{X}^{\mathsf{mult}} \mid \left\{ \begin{array}{l} \left. \begin{array}{l} \exists v \in H^1(\Omega) \\ \text{s.t. } \forall j \in \{1,2\} \end{array} \right\} : \quad \psi_{\mathsf{D};j} = \gamma_{\mathsf{D};j} v \\ \left. \begin{array}{l} \exists \mathbf{w} \in \mathbf{H}(\Omega, \mathsf{div}) \\ \text{s.t. } \forall j \in \{1,2\} \end{array} \right\} : \quad \psi_{\mathsf{N};j} = \left\langle \mathbf{w}, \mathbf{n}_j \right\rangle \end{array} \right. \right\},$$

$$\mathbf{X}_0^{\mathsf{single}} := \left\{ \psi \in \mathbf{X}^{\mathsf{single}} \mid \forall j \in \{1,2\} : \left\{ \begin{array}{rl} & \psi_{\mathsf{D};j}\big|_{\Gamma_j \cap \Gamma_{\mathsf{D}}} = 0 \\ \wedge & \psi_{\mathsf{N};j}\big|_{\Gamma_j \cap \Gamma_{\mathsf{N}}} = 0 \end{array} \right. \right\}.$$

Set $\mathbf{u}^{\mathsf{single}} := \left(\mathbf{u}_j^{\mathsf{mult}} - \boldsymbol{\beta}_j\right)_{j=1}^{n_\Omega}$ and observe that $\mathbf{u}^{\mathsf{single}}$ satisfies

$$\left(\mathsf{C}_j(s) - \tfrac{1}{2}\,\mathsf{Id}_j\right)\mathbf{u}_j^{\mathsf{single}} = -\left(\mathsf{C}_j(s) - \tfrac{1}{2}\,\mathsf{Id}_j\right)\boldsymbol{\beta}_j \quad \text{in } \Omega_j \quad j = 1, 2,$$

$$\left[\mathbf{u}^{\mathsf{single}}\right]_{1,2} = 0$$

$$\left.\begin{array}{l} \left.u_{\mathsf{D};j}^{\mathsf{single}}\right|_{\Gamma_j \cap \Gamma_{\mathsf{D}}} = 0 \\ \left.u_{\mathsf{N};j}^{\mathsf{single}}\right|_{\Gamma_j \cap \Gamma_{\mathsf{N}}} = 0 \end{array}\right\} \qquad j = 1, 2.$$

This implies that $\mathbf{u}^{\mathsf{single}} \in \mathbf{X}_0^{\mathsf{single}}$.

Variational form of the non-local skeleton problem in the single trace space:

Find $\mathbf{u}^{\mathsf{single}} \in \mathbf{X}_0^{\mathsf{single}}$ such that

$$c(s)\left(\mathbf{u}^{\mathsf{single}}, \boldsymbol{\psi}\right) = -c(s)(\boldsymbol{\beta}, \boldsymbol{\psi}) \quad \forall \boldsymbol{\psi} \in \mathbf{X}_0^{\mathsf{single}}.$$

Set $\mathbf{u}_j^{\mathsf{mult}} := \mathbf{u}^{\mathsf{single}} + \boldsymbol{\beta}$ so that Green's representation formula yields

$$u_j := \left(\mathsf{S}_j(s)\, u_{\mathsf{N};j}^{\mathsf{mult}} - \mathsf{D}_j(s)\, u_{\mathsf{D};j}^{\mathsf{mult}}\right)\Big|_{\Omega_j} \quad j = 1, 2.$$

The function $\mathbf{u} = \left(u_j\right)_{j=1}^{n_\Omega} \in \mathbf{H}(\Omega, \mathbb{A})$ finally solves the original transmission problem.

Frequency explicit coercivity and continuity estimates:

Lemma (Florian, Hiptmair, STAS, 2022). The layer potentials and skeleton operators satisfy the coercivity and continuity estimates:

Continuity of layer potentials:

$$\left|\left|\left|\mathsf{S}_j(s)\,\varphi\right|\right|\right|_{\mathbb{R}^3;s} \leq C\frac{|s|^{3/2}}{\lambda\operatorname{Re}s}\|\varphi\|_{H^{-1/2}(\Gamma_j)} \qquad \forall\varphi\in H^{-1/2}\left(\Gamma_j\right),$$

$$\left\|\mathsf{D}_j(s)\,\psi\right\|_{H^1(\mathbb{R}^3\backslash\Gamma_j);s} \leq C\frac{\Lambda}{\lambda}\frac{|s|}{\operatorname{Re}s}\|\psi\|_{H^{1/2}(\Gamma_j)} \qquad \forall\psi\in H^{1/2}\left(\Gamma_j\right).$$

Coercivity of skeleton operators:

$$\mathsf{Re}\left\langle \varphi, \overline{\mathsf{V}_j(s)\,\varphi} \right\rangle_{\Gamma_j} \geq c\frac{\mathsf{Re}\,s}{|s|}\frac{\lambda}{\Lambda^2}\,\|\varphi\|^2_{H^{-1/2}(\Gamma_j)} \qquad \forall \varphi \in H^{-1/2}\left(\Gamma_j\right),$$

$$\mathsf{Re}\left\langle \mathsf{W}_j(s)\,\psi, \overline{\psi} \right\rangle_{\Gamma_j} \geq c\frac{\mathsf{Re}\,s}{|s|^2}\lambda\,\|\psi\|^2_{H^{1/2}(\Gamma_j)} \qquad \forall \psi \in H^{1/2}\left(\Gamma_j\right),$$

Continuity of skeleton operators:

$$\left|\left\langle \mathsf{V}_j(s)\,\varphi, \overline{\psi} \right\rangle_{\Gamma_j}\right| \leq C\frac{|s|^2}{\lambda\,\mathsf{Re}\,s}\,\|\varphi\|_{H^{-1/2}(\Gamma_j)}\,\|\psi\|_{H^{-1/2}(\Gamma_j)} \qquad \forall \varphi,\psi \in H^{-1/2}\left(\Gamma_j\right),$$

$$\left\|\mathsf{K}'_j(s)\,\varphi\right\|_{H^{-1/2}(\Gamma_j)} \leq C\frac{\Lambda}{\lambda}\frac{|s|^{3/2}}{\mathsf{Re}\,s}\,\|\varphi\|_{H^{-1/2}(\Gamma_j)} \qquad \forall \varphi \in H^{-1/2}\left(\Gamma_j\right),$$

$$\left\|\mathsf{K}_j(s)\,\psi\right\|_{H^{1/2}(\Gamma_j)} \leq C\frac{\Lambda}{\lambda}\frac{|s|^{3/2}}{\mathsf{Re}\,s}\,\|\psi\|_{H^{1/2}(\Gamma_j)} \qquad \forall \psi \in H^{1/2}\left(\Gamma_j\right),$$

$$\left|\left\langle \mathsf{W}_j(s)\,\psi, \overline{\varphi} \right\rangle_{\Gamma_j}\right| \leq C\frac{\Lambda^2}{\lambda}\frac{|s|}{\mathsf{Re}\,s}\,\|\psi\|_{H^{1/2}(\Gamma_j)}\,\|\varphi\|_{H^{1/2}(\Gamma_j)} \qquad \forall \varphi,\psi \in H^{1/2}\left(\Gamma_j\right).$$

Well-posedness of single-trace formulation:

Theorem.

a) The sesquilinear form $c(s)(\cdot,\cdot) : \mathbf{X}_0^{\text{single}} \times \mathbf{X}_0^{\text{single}} \longrightarrow \mathbb{C}$ is coercive and continuous: for any $\boldsymbol{\alpha} \in \mathbf{X}_0^{\text{single}}$ and $\boldsymbol{\psi}, \boldsymbol{\phi} \in \mathbf{X}^{\text{mult}}$ it holds

$$\operatorname{Re} c(s)(\boldsymbol{\alpha},\boldsymbol{\alpha}) \geq c \frac{\lambda}{1+\Lambda^2} \frac{\operatorname{Re} s}{|s|^2} \, \|\boldsymbol{\alpha}\|_{\mathbb{X}}^2 \,,$$

$$c(s)(\boldsymbol{\psi},\boldsymbol{\phi}) \leq \left(\frac{1}{2} + C \frac{1+\Lambda}{\lambda} \frac{|s|^2}{\operatorname{Re} s} \right) \|\boldsymbol{\psi}\|_{\mathbb{X}} \, \|\boldsymbol{\phi}\|_{\mathbb{X}} \,.$$

b) For any $\boldsymbol{\beta} \in \mathbf{X}^{\mathsf{mult}}$, the variational skeleton problem has a solution $\mathbf{u}^{\mathsf{single}} \in \mathbf{X}_0^{\mathsf{single}}$ which is unique and satisfies

$$\left\| \mathbf{u}^{\mathsf{single}} \right\|_{\mathbb{X}} \leq C \frac{|s|^4}{(\mathsf{Re}\, s)^2} \left\| \boldsymbol{\beta} \right\|_{\mathbb{X}} .$$

Proof: Florian, Hiptmair, STAS, 2022.

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All my best wishes,

Ernst,

for the future!