Recent Advances in Elastodynamics by Time-Domain Energetic Boundary Element Method

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Elastodynamic Model Problem

$\Omega \subset \mathbb{R}^n \qquad n = 2, 3$ Navier Equations

$$\sum_{h,k,l=1}^{n} \frac{\partial}{\partial x_h}\left(\mathcal{C}_{ih}^{kl}\frac{\partial u_k}{\partial x_l}(\mathbf{x},t)\right) - \rho\ddot{u}_i(\mathbf{x},t) = 0 \qquad i = 1,\dots,n \qquad \forall(\mathbf{x},t) \in \Omega \times (0,T]$$

$$\mathbf{u}(\mathbf{x},0) = 0; \quad \dot{\mathbf{u}}(\mathbf{x},0) = 0 \qquad \mathbf{x} \in \Omega$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{g}_D(\mathbf{x},t) \qquad \mathbf{x} \in \Sigma_D := \Gamma_D \times [0,T]$$

$$\mathbf{p}(\mathbf{x},t) = \mathbf{g}_N(\mathbf{x},t) \qquad \mathbf{x} \in \Sigma_N := \Gamma_N \times [0,T]$$

$$\mathcal{C}_{ih}^{kl} := \lambda\delta_{ih}\delta_{kl} + \mu(\delta_{ik}\delta_{hl} + \delta_{il}\delta_{hk}), \qquad h,k,l = 1,\dots,n$$ Hooke tensor

λ, μ Lamè constants incorporating the physical nature of stress-strain relationship involved in the elastic medium

ρ mass density

$$p_i(\mathbf{x},t) := \sum_{h,k,l=1}^{n} \mathcal{C}_{ih}^{kl}\frac{\partial u_k}{\partial x_l}(\mathbf{x},t)n_h(\mathbf{x}), \qquad i = 1,\dots,n$$ Traction vector

Components of outward normal vector

- The displacement $\mathbf{u}(\mathbf{x},\mathbf{t})$ may be decomposed into an irrotational (or dilatational) part $\mathbf{u}_P(\mathbf{x},\mathbf{t})$ and a rotational (or equivoluminal) part $\mathbf{u}_S(\mathbf{x},\mathbf{t})$

$$\boxed{\mathbf{u} = \mathbf{u}_P + \mathbf{u}_S = \nabla\phi + \nabla \wedge \boldsymbol{\psi}}$$

satisfying the scalar and the vector wave equations

$$c_P^2 \Delta\phi - \ddot{\phi} = 0 \qquad\qquad c_S^2 \boldsymbol{\Delta}\boldsymbol{\psi} - \ddot{\boldsymbol{\psi}} = 0$$

- The first one propagates with the phase speed

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad > \quad c_S = \sqrt{\frac{\mu}{\rho}}$$

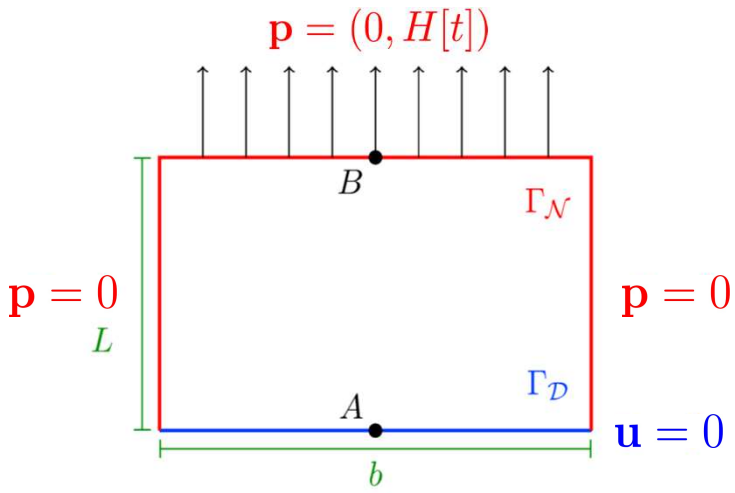
PRIMARY waves **SECONDARY waves**

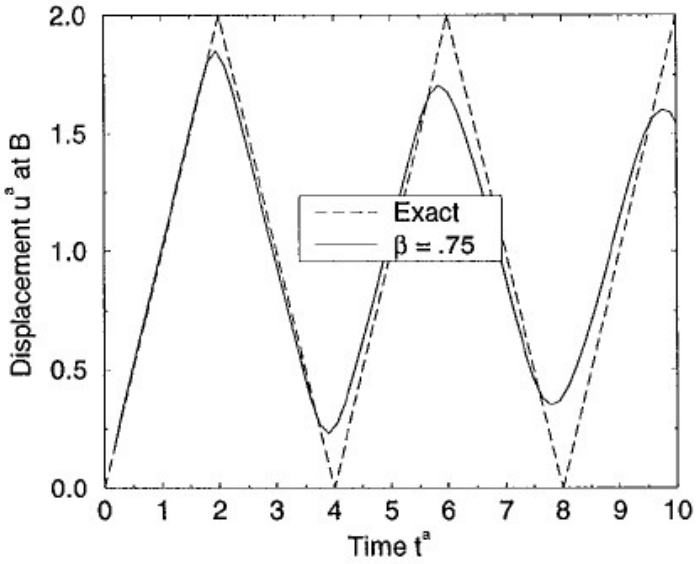
Motivation: common thread

BEM applied to Galerkin L_2 BIE weak formulation directly in space-time domain can be unstable

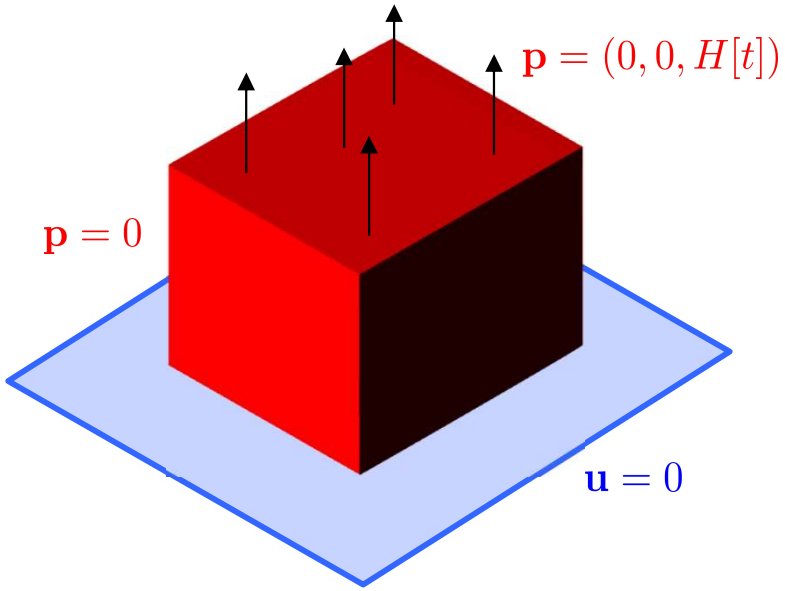
Frangi A. and Novati G.
Comput. Methods Appl. Mech. Engrg. (1999)
On the numerical stability of time-domain elastodynamic analyses by BEM.

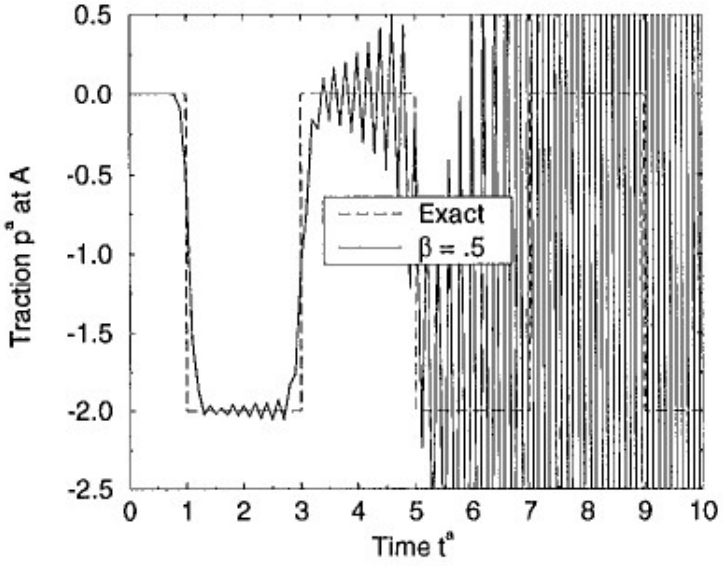
$n = 2$





$n = 3$





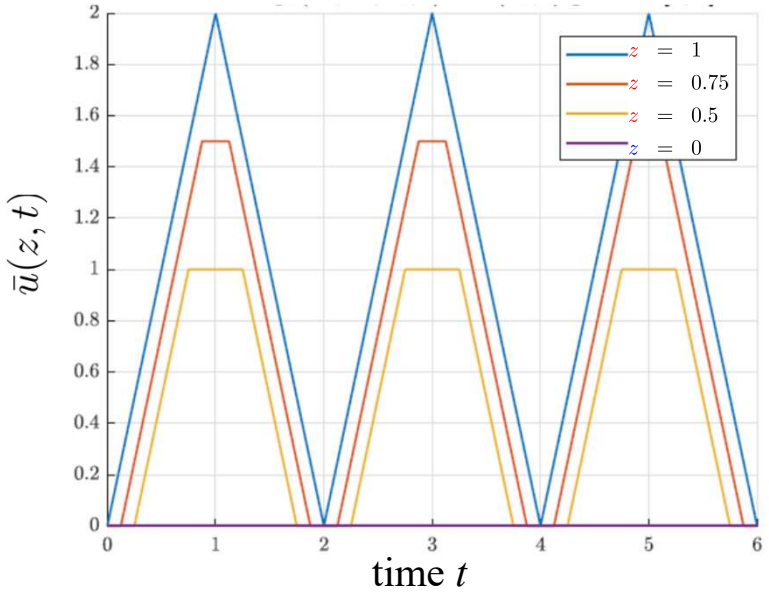
Assuming null strain in the direction orthogonal to that of stress application:

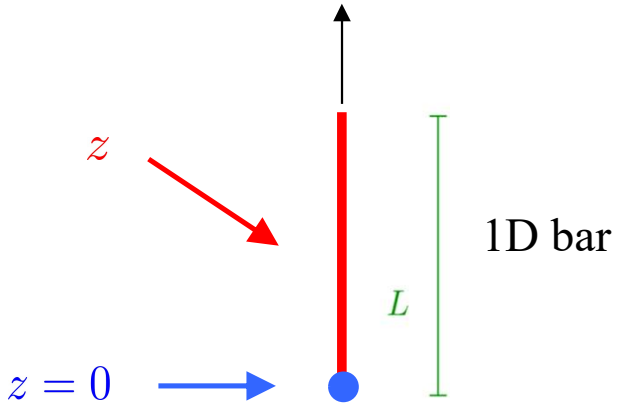
$$\lambda = 0 \quad \rightarrow \quad c_P = \sqrt{2} c_S \quad \rightarrow \quad \text{1D reference solution}$$

Displacement only in vertical direction

$$\bar{u}(z,t) = \frac{H[t]}{\rho c_P^2} \sum_{k=0}^{\lceil c_p T/(2L) \rceil} (-1)^k \left\{ \left(c_p t - 2kL - (L-z)\right) H\left[\frac{c_p t - 2kL - (L-z)}{c_p}\right] - \left(c_p t - 2(k+1)L - (L-z)\right) H\left[\frac{c_p t - 2(k+1)L - (L-z)}{c_p}\right] \right\}$$

$z \in [0, L] \quad t \in [0, T]$





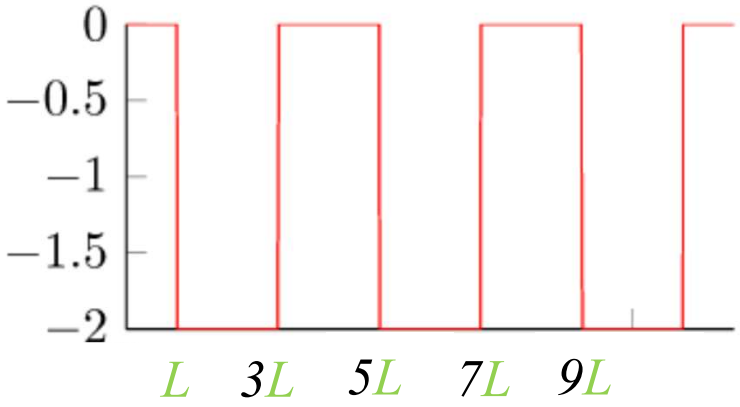
L length of the bar, z coordinate in vertical direction

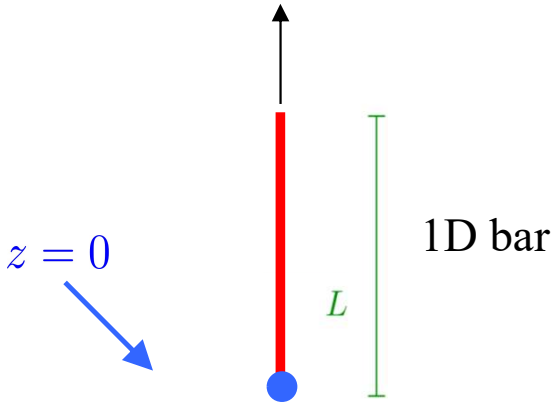
Assuming null strain in the direction orthogonal to that of stress application:

$$\lambda = 0 \quad \rightarrow \quad c_P = \sqrt{2} c_S \quad \rightarrow \quad \text{1D reference solution}$$

Traction at the fixed bottom edge

$$\bar{p}(0,t) = \frac{H[t]}{\rho c_P^2} \sum_{k=0}^{\lceil c_p T/(2L) \rceil} (-1)^k \left\{ H\left[\frac{c_p t - (2k+1)L}{c_p}\right] - H\left[\frac{c_p t - (2k+3)L}{c_p}\right] \right\} \qquad t \in [0,T]$$





L length of the bar

Complete Integral Representation Formula $(\mathbf{x},t)\in\Omega\times(0,T]$ $i=1,\dots,n$

$$u_i(\mathbf{x},t)=(\mathcal{V}\,\mathbf{p})_i(\mathbf{x},t)-(\mathcal{K}\,\mathbf{u})_i(\mathbf{x},t)$$

$$(\mathcal{V}\mathbf{p})_i(\mathbf{x},t):=\sum_{j=1}^{n}\int_0^t\int_\Gamma G_{ij}(\mathbf{x},\mathbf{y};t,\tau)\,p_j(\mathbf{y},\tau)d\Gamma_{\mathbf{y}}d\tau$$

Single layer operator

$$(\mathcal{K}\mathbf{u})_i(\mathbf{x},t):=\sum_{j=1}^{n}\int_0^t\int_\Gamma\sum_{h,k,l=1}^{n}\mathcal{C}_{jh}^{kl}\frac{\partial G_{ik}}{\partial y_l}(\mathbf{x},\mathbf{y};t,\tau)\,u_j(\mathbf{y},\tau)d\Gamma_{\mathbf{y}}d\tau$$

Double layer operator

Complete Integral Representation Formula $(\mathbf{x},t)\in\Omega\times(0,T]$ $i=1,2$

$$u_i(\mathbf{x},t)=(\mathcal{V}\,\mathbf{p})_i(\mathbf{x},t)-(\mathcal{K}\,\mathbf{u})_i(\mathbf{x},t)$$

$$(\mathcal{V}\mathbf{p})_i(\mathbf{x},t):=\sum_{j=1}^{2}\int_0^t\int_\Gamma G_{ij}(\mathbf{x},\mathbf{y};t,\tau)\,p_j(\mathbf{y},\tau)d\Gamma_{\mathbf{y}}d\tau$$ Single layer operator

$$(\mathcal{K}\mathbf{u})_i(\mathbf{x},t):=\sum_{j=1}^{2}\int_0^t\int_\Gamma\sum_{h,k,l=1}^{2}C_{jh}^{kl}\frac{\partial G_{ik}}{\partial y_l}(\mathbf{x},\mathbf{y};t,\tau)\,u_j(\mathbf{y},\tau)d\Gamma_{\mathbf{y}}d\tau$$ Double layer operator

2D Time-domain Green's function $i=1,2$

$$\begin{aligned}G_{ij}(\mathbf{x},\boldsymbol{\xi};t,\tau) \;:=\;& \frac{H\left[c_P(t-\tau)-r\right]}{2\pi\rho c_P}\left\{\frac{r_ir_j}{r^4}\frac{2c_P^2(t-\tau)^2-r^2}{\sqrt{c_P^2(t-\tau)^2-r^2}}-\frac{\delta_{ij}}{r^2}\sqrt{c_P^2(t-\tau)^2-r^2}\right\}\\ -\;& \frac{H\left[c_S(t-\tau)-r\right]}{2\pi\rho c_S}\left\{\frac{r_ir_j}{r^4}\frac{2c_S^2(t-\tau)^2-r^2}{\sqrt{c_S^2(t-\tau)^2-r^2}}-\frac{\delta_{ij}}{r^2}\frac{c_S^2(t-\tau)^2}{\sqrt{c_S^2(t-\tau)^2-r^2}}\right\}\end{aligned}$$

$$\mathbf{r}:=\mathbf{x}-\boldsymbol{\xi}=(r_1,r_2)=(x_1-y_1,x_2-y_2)\qquad r=|\mathbf{r}|=\sqrt{r_1^2+r_2^2}$$

Complete Integral Representation Formula $(\mathbf{x},t)\in\Omega\times(0,T]$ $i=1,2,3$

$$u_i(\mathbf{x},t)=(\mathcal{V}\,\mathbf{p})_i(\mathbf{x},t)-(\mathcal{K}\,\mathbf{u})_i(\mathbf{x},t)$$

$$(\mathcal{V}\mathbf{p})_i(\mathbf{x},t):=\sum_{j=1}^{3}\int_0^t\int_\Gamma G_{ij}(\mathbf{x},\mathbf{y};t,\tau)\,p_j(\mathbf{y},\tau)d\Gamma_{\mathbf{y}}d\tau$$ Single layer operator

$$(\mathcal{K}\mathbf{u})_i(\mathbf{x},t):=\sum_{j=1}^{3}\int_0^t\int_\Gamma\sum_{h,k,l=1}^{3}C_{jh}^{kl}\frac{\partial G_{ik}}{\partial y_l}(\mathbf{x},\mathbf{y};t,\tau)\,u_j(\mathbf{y},\tau)d\Gamma_{\mathbf{y}}d\tau$$ Double layer operator

3D Time-domain Green's function $i=1,2,3$

$$\begin{aligned}G_{ij}(\mathbf{x},\boldsymbol{\xi};t,\tau) \;:=\;& \frac{1}{4\pi\rho c_P^2}\frac{r_ir_j}{r^3}\,\delta\left[t-\tau-\frac{r}{c_P}\right]+\frac{1}{4\pi\rho c_S^2}\left(\frac{\delta_{ij}}{r}-\frac{r_ir_j}{r^3}\right)\delta\left[t-\tau-\frac{r}{c_S}\right]\\ -\;& \frac{1}{4\pi\rho}\left(\frac{\delta_{ij}}{r^3}-3\frac{r_ir_j}{r^5}\right)(t-\tau)\left(H\left[t-\tau-\frac{r}{c_P}\right]-H\left[t-\tau-\frac{r}{c_S}\right]\right)\end{aligned}$$

$$\mathbf{r}:=\mathbf{x}-\boldsymbol{\xi}=(r_1,r_2,r_3)=(x_1-y_1,x_2-y_2,x_3-y_3)\qquad r=|\mathbf{r}|=\sqrt{r_1^2+r_2^2+r_3^2}$$

Complete Boundary Integral Equation $\quad (\mathbf{x},t) \in \Gamma \times (0,T] \qquad i = 1,\dots,n$

$$\frac{1}{2} u_i(\mathbf{x},t) = (\mathcal{V}\,\mathbf{p})_i(\mathbf{x},t) - (\mathcal{K}\,\mathbf{u})_i(\mathbf{x},t)$$

$$\frac{1}{2}\begin{pmatrix} u_1(\mathbf{x},t) \\ u_2(\mathbf{x},t) \end{pmatrix} = \begin{pmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22} \end{pmatrix} \begin{pmatrix} p_1(\mathbf{x},t) \\ p_2(\mathbf{x},t) \end{pmatrix} - \begin{pmatrix} \mathcal{K}^{11} & \mathcal{K}^{12} \\ \mathcal{K}^{21} & \mathcal{K}^{22} \end{pmatrix} \begin{pmatrix} u_1(\mathbf{x},t) \\ u_2(\mathbf{x},t) \end{pmatrix}$$

Energy relation

$$\mathcal{E}(\mathbf{u},T) := \frac{1}{2}\sum_{i=1}^{n}\left\{ \int_\Omega \rho \dot{u}_i(\mathrm{x},T)d\mathrm{x} + \int_\Omega \sum_{h,k,l=1}^{n} \mathcal{C}_{ih}^{kl} \frac{\partial u_k}{\partial y_l}(\mathbf{x},T)\frac{\partial u_h}{\partial y_l}(\mathbf{x},T)d\mathbf{x} \right\} = \sum_{i=1}^{n}\int_0^T \int_\Gamma p_i(\mathbf{x},t)\dot{u}_i(\mathbf{x},t)d\mathbf{x}dt$$

Time-domain Energetic weak formulation $\qquad i = 1,\dots,n$

$$\frac{1}{2}\left\langle \dot{u}_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle (\dot{\mathcal{V}}\,\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle (\dot{\mathcal{K}}\,\mathbf{u})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)}$$

$$u_i \in H^1\big([0,T]; H^{\frac{1}{2}}(\Gamma)\big) \qquad p_i, \phi_i \in L^2\big([0,T]; H^{-\frac{1}{2}}(\Gamma)\big), \qquad i = 1,\dots,n$$

An energy approach to space–time Galerkin BEM for wave propagation problems, IJNME, (2009)

$$
\begin{aligned}
u_i(\mathbf{x},t) &\approx \sum_{n=0}^{N_{\Delta t}-1}\sum_{m=1}^{M_{\mathbf{u}}} \alpha_{nm}^{\mathbf{u},i} w_m^{\mathbf{u}}(\mathbf{x}) r_n(t) \\
p_i(\mathbf{x},t) &\approx \sum_{n=0}^{N_{\Delta t}-1}\sum_{m=1}^{M_{\mathbf{p}}} \alpha_{nm}^{\mathbf{p},i} w_m^{\mathbf{p}}(\mathbf{x}) v_n(t)
\end{aligned}
\qquad i = 1,\ldots,n
$$

- Uniform decomposition of the time interval $\Delta t = T/N_{\Delta t} \quad N_{\Delta t} \in \mathbb{N}^+, \quad t_n = n\Delta t, \quad n = 0,\ldots,N_{\Delta t}$

$V_{\Delta t,1}$

continuous piecewise linear basis functions for displacements

$$r_n(t) := \frac{t-t_n}{\Delta t} H[t-t_n] - \frac{t-t_{n+1}}{\Delta t} H[t-t_{n+1}]$$

$V_{\Delta t,0}$

piecewise constant basis functions for tractions

$$v_n(t) := H[t-t_n] - H[t-t_{n+1}]$$

- Uniform decomposition of boundary $\Gamma = \bigcup_{i=1}^{M} e_i \qquad e_i \cap e_j = \emptyset,\ i \neq j \qquad \operatorname{diam}(e_i) = \Delta x$

piecewise polynomial boundary element basis functions of degree *d*

$w_m^{\mathbf{u}}(\mathbf{x}) \in X_{\Delta x,d}^{0} := \{w \in C^0(\Gamma) : w_{|e_i} \in \mathcal{P}_d,\ i = 1,\ldots,M\}$

$w_m^{\mathbf{p}}(\mathbf{x}) \in X_{\Delta x,d}^{-1} := \{w \in L^2(\Gamma) : w_{|e_i} \in \mathcal{P}_d,\ i = 1,\ldots,M\}$

Implementation of EBEM is not straightforward:
the computation of linear system entries depends on the problem at hand and on the used integral formulation

3D Dirichlet problem $u = \bar{u}$: indirect formulation

Find $\varphi \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^3$ such that

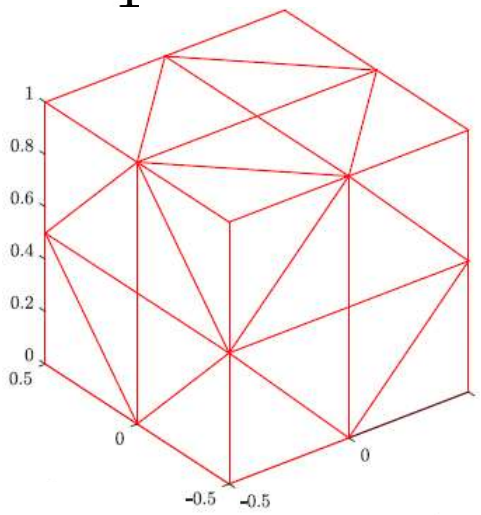
$$\left\langle (\dot{\mathcal{V}}\varphi)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \dot{\bar{u}}_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \qquad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^3$$

$\rho = 1/4$
$c_P = 2$
$c_S = \sqrt{2}$

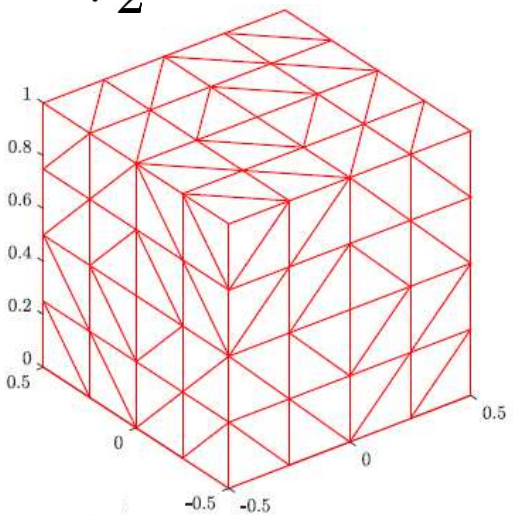
τ_i	Δx	M	Δt
$i = 1$	0.5000	48	0.25000
$i = 2$	0.2500	192	0.12500
$i = 3$	0.1250	768	0.06250
$i = 4$	0.0625	3072	0.03125

$$\frac{\Delta x}{\Delta t} = c_P = 2$$

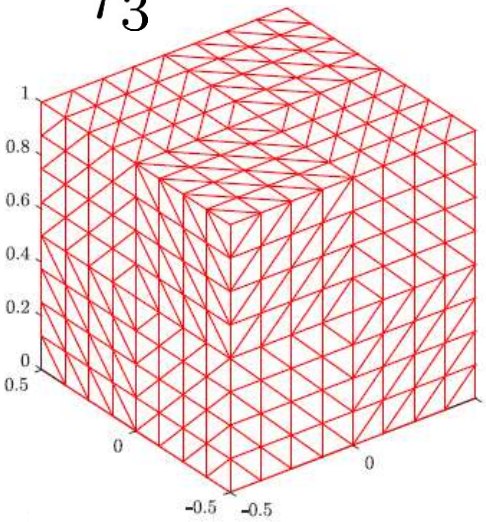
τ_1



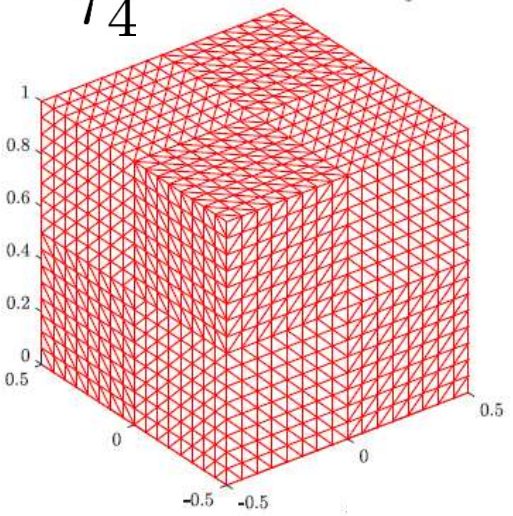
τ_2



τ_3



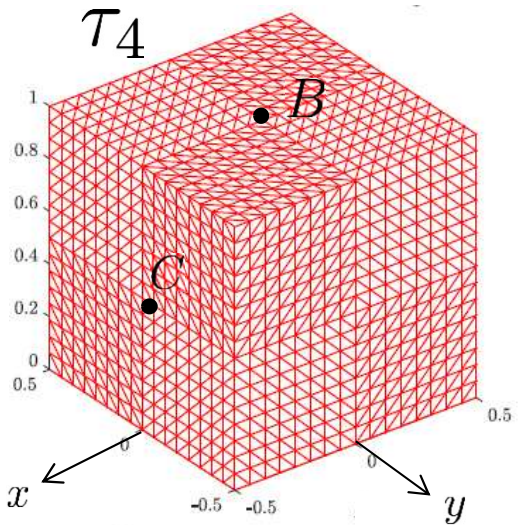
τ_4



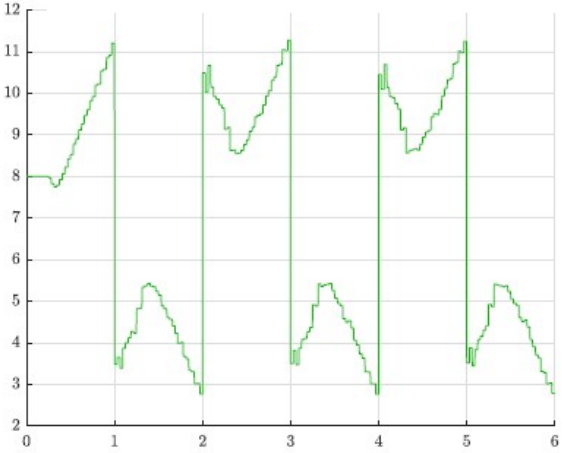
3D Dirichlet problem $u = \bar{u}$: indirect formulation

Find $\varphi \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^3$ such that

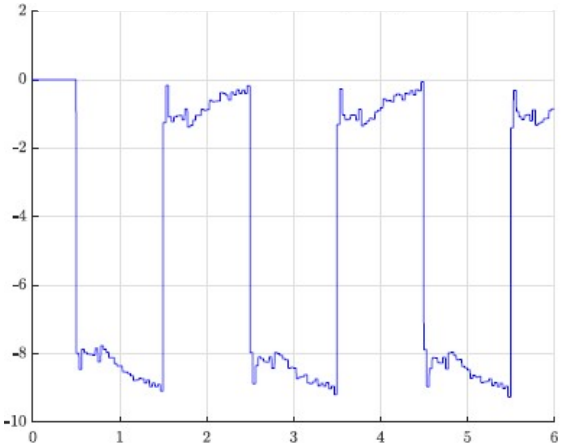
$$\left\langle (\dot{\mathcal{V}}\varphi)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \dot{\bar{u}}_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \qquad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^3$$



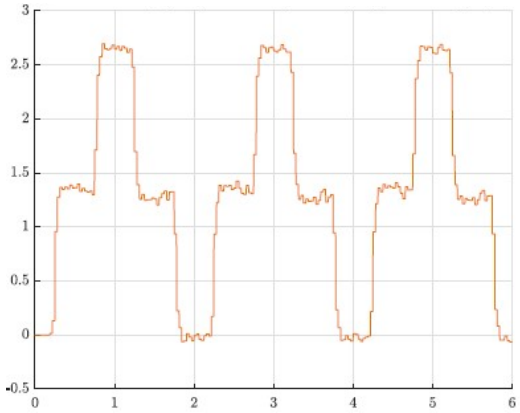
$\phi_3(B,t) \quad B \approx (0,0,1)$



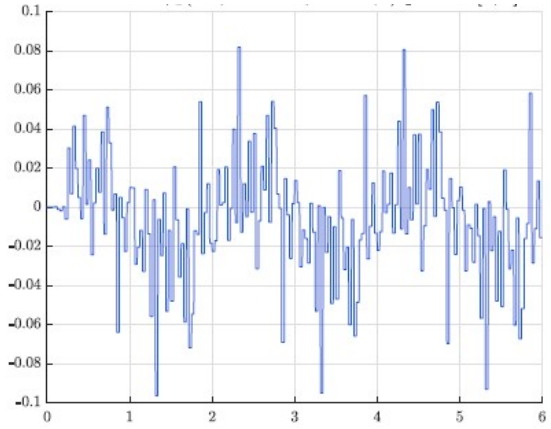
$\phi_3(A,t) \quad A \approx (0,0,0)$



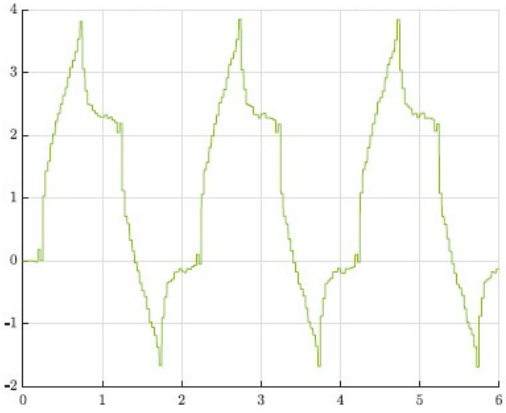
$\phi_1(C,t) \quad C \approx (0.5,0,0.5)$



$\phi_2(C,t)$



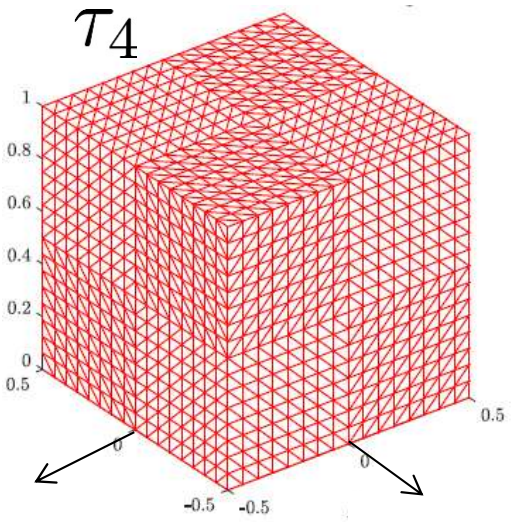
$\phi_3(C,t)$



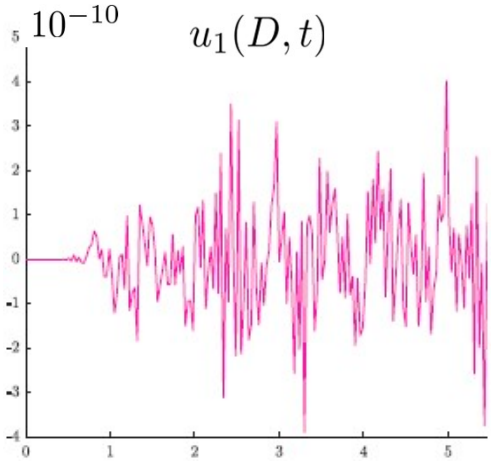
3D Dirichlet problem $u = \bar{u}$: indirect formulation

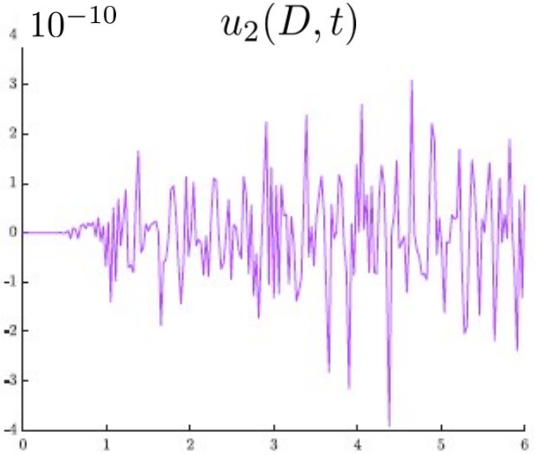
Find $\varphi \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^3$ such that

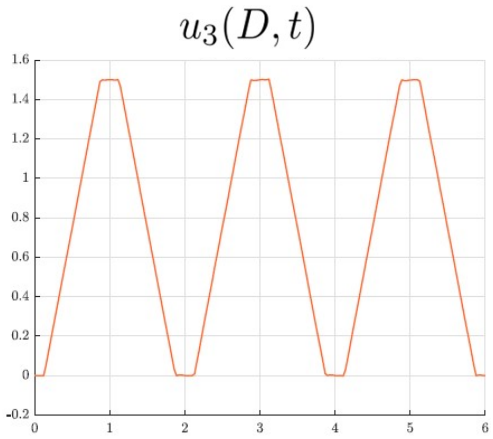
$$\left\langle (\dot{\mathcal{V}}\varphi)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \dot{\bar{u}}_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \qquad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^3$$



Post-processing inside the domain at $D \approx (0, 0, 0.75)$







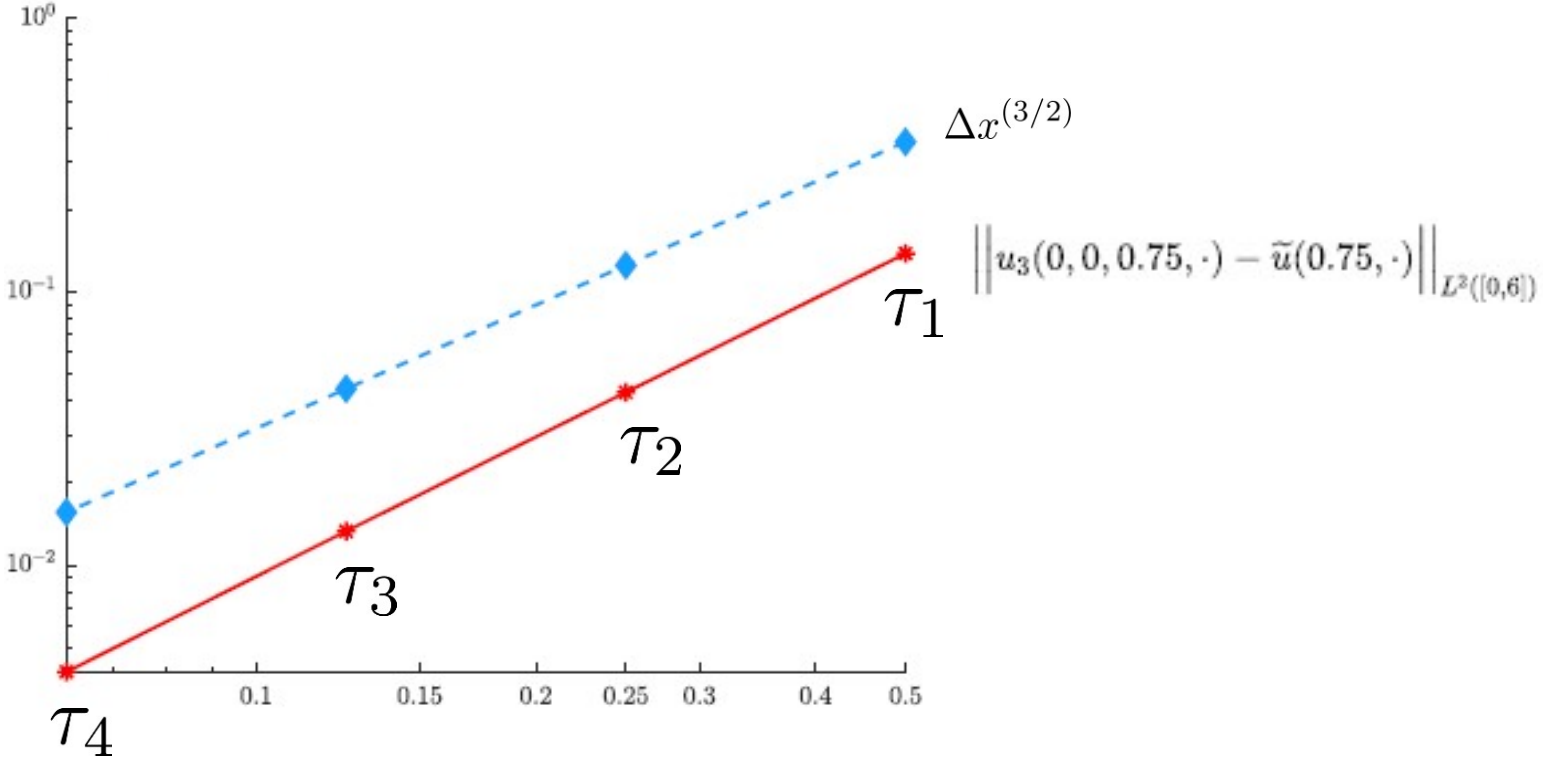
Single Layer approach in Energetic BEM approximation of 3D Elastodynamics, in preparation

3D Dirichlet problem $u = \bar{u}$: indirect formulation

Find $\boldsymbol{\varphi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\boldsymbol{\varphi})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \dot{\bar{u}}_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \qquad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^2$$

Convergence

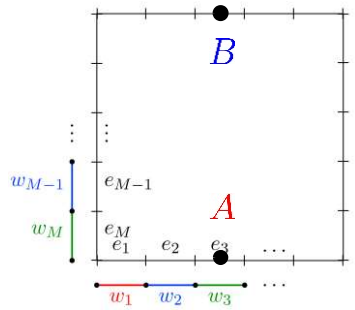


2D Dirichlet problem $\mathbf{u} = \bar{\mathbf{u}}$: direct formulation, strong imposition of datum

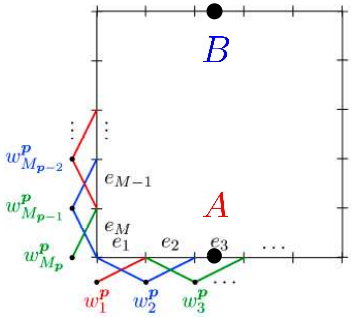
$\rho = 1$
$c_P = 1$
$c_S = \dfrac{1}{\sqrt{2}}$
$L = 1$

Find $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^2$ such that

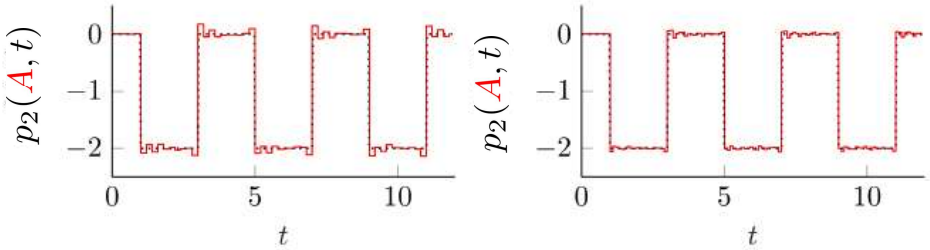
$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\bar{\mathbf{u}}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \qquad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

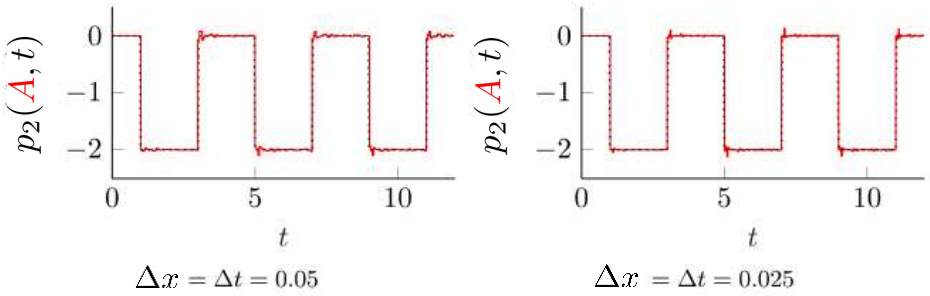


$$\frac{\Delta x}{\Delta t} = c_P = 1$$

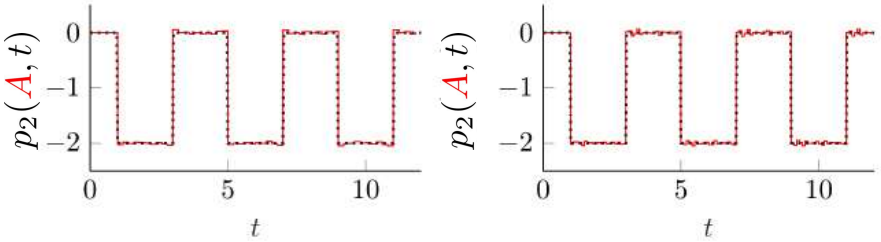


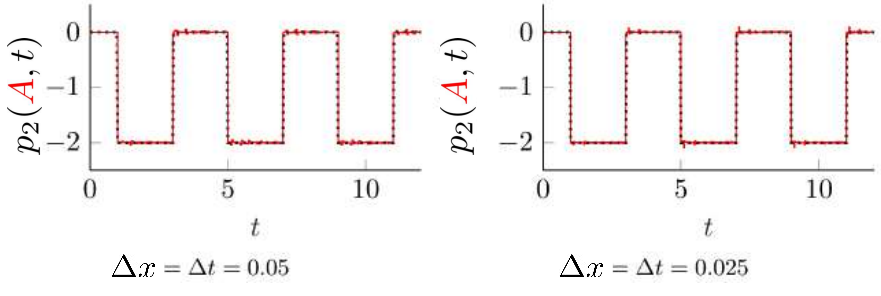
$d = \bar{d} = 0$ piecewise constant basis functions in space





$d = \bar{d} = 1$ piecewise linear basis functions in space



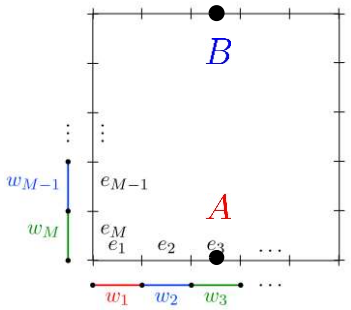


2D Dirichlet problem $\mathbf{u} = \bar{\mathbf{u}}$: direct formulation, strong imposition of datum

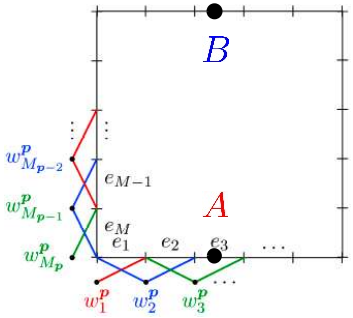
$\rho = 1$
$c_P = 1$
$c_S = \dfrac{1}{\sqrt{2}}$
$L = 1$

Find $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,d}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\bar{\mathbf{u}}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \qquad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$



$$\frac{\Delta x}{\Delta t} = c_P = 1$$



$d = \bar{d} = 0$ piecewise constant basis functions in space

$d = \bar{d} = 1$ piecewise linear basis functions in space

Δx	E_1	$rate_1$	E_2	$rate_2$
0.2	0.5347	-	0.4612	-
0.1	0.3705	0.5292	0.3122	0.5630
0.05	0.2574	0.5255	0.2161	0.5306
0.025	0.1793	0.5217	0.1488	0.5388

Δx	E_1	$rate_1$	E_2	$rate_2$
0.2	0.2567	-	0.3117	-
0.1	0.1465	0.8097	0.1742	0.8392
0.05	0.0930	0.6545	0.1039	0.7454
0.025	0.0608	0.6129	0.0621	0.7439

$$E_i = \|\bar{p}_i - p_i\|_{L^2(\Sigma)} \qquad rate_i^k = \log_2\left(\frac{E_i^{k-1}}{E_i^k}\right)$$

2D Dirichlet problem $\mathbf{u} = \bar{\mathbf{u}}$: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

$$\left\langle \dot{u}_i(\mathbf{x},t), \psi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \dot{\bar{u}}_i(\mathbf{x},t), \psi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$$

Same numerical results for traction as before and good reconstruction of Dirichlet datum(10^{-13})

unknowns $\quad \boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p}}+2M_{\mathbf{u}}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$

Toeplitz matrix

$$\begin{pmatrix} \mathbb{E}^{(0)} & 0 & 0 & \cdots & 0 \\ \mathbb{E}^{(1)} & \mathbb{E}^{(0)} & 0 & \cdots & 0 \\ \mathbb{E}^{(2)} & \mathbb{E}^{(1)} & \mathbb{E}^{(0)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}^{(N_{\Delta t-1})} & \mathbb{E}^{(N_{\Delta t-2})} & \mathbb{E}^{(N_{\Delta t-3})} & \cdots & \mathbb{E}^{(0)} \end{pmatrix}$$

$$\mathbb{E}^{(0)} = \begin{pmatrix} \mathbb{E}^{(0)}_{\mathcal{V}} & -\mathbb{E}^{(0)}_{\mathcal{K}} - \frac{1}{2}\mathbb{M}_{\mathbf{pu}} \\ \mathbb{O}_{2M_{\mathbf{u}} \times 2M_{\mathbf{p}}} & \frac{1}{2}\mathbb{M}_{\mathbf{uu}} \end{pmatrix}$$

$$\mathbb{E}^{(\ell)} = \begin{pmatrix} \mathbb{E}^{(\ell)}_{\mathcal{V}} & -\mathbb{E}^{(\ell)}_{\mathcal{K}} \\ \mathbb{O}_{2M_{\mathbf{u}} \times 2M_{\mathbf{p}}} & \mathbb{M}_{\mathbf{uu}} \end{pmatrix}$$

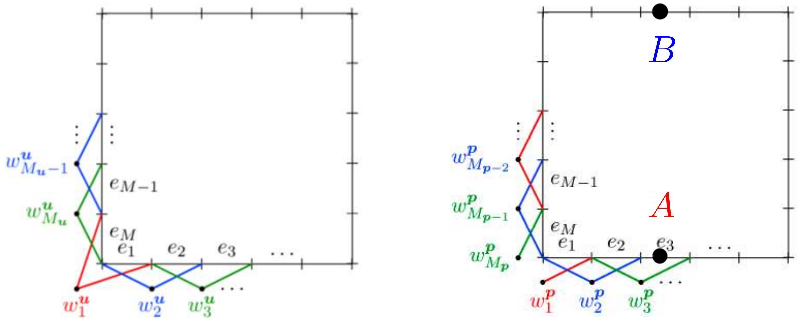
Blocks must be squared

2D Neumann problem $\dot{\mathbf{p}} = \bar{\dot{\mathbf{p}}}$: direct formulation, strong imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^0_{\Delta x,d}\right]^2$ such that

$$\left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle (\mathcal{V}\dot{\mathbf{p}})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

Classical direct formulation needs same approximation space for basis and test functions



2D Neumann problem $\mathbf{p} = \bar{\mathbf{p}}$: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in [V_{\Delta t,0} \times X^{*}]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \bar{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\psi} \in [V_{\Delta t,1} \times X^{\star}]^2$$

unknowns $\quad \boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p}}+2M_{\mathbf{u}}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$

Toeplitz matrix

$$\begin{pmatrix} \mathbb{E}^{(0)} & 0 & 0 & \cdots & 0 \\ \mathbb{E}^{(1)} & \mathbb{E}^{(0)} & 0 & \cdots & 0 \\ \mathbb{E}^{(2)} & \mathbb{E}^{(1)} & \mathbb{E}^{(0)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}^{(N_{\Delta t-1})} & \mathbb{E}^{(N_{\Delta t-2})} & \mathbb{E}^{(N_{\Delta t-3})} & \cdots & \mathbb{E}^{(0)} \end{pmatrix}$$

$$\mathbb{E}^{(0)} = \begin{pmatrix} \mathbb{E}^{(0)}_{\mathcal{V}} & -\mathbb{E}^{(0)}_{\mathcal{K}} - \frac{1}{2}\mathbb{M}_{*\mathbf{u}} \\ \frac{1}{2}\mathbb{M}_{\star\mathbf{p}} & \mathbb{O}_{2M_\star \times 2M_{\mathbf{u}}} \end{pmatrix}$$

$$\mathbb{E}^{(\ell)} = \begin{pmatrix} \mathbb{E}^{(\ell)}_{\mathcal{V}} & -\mathbb{E}^{(\ell)}_{\mathcal{K}} \\ \mathbb{M}_{\star\mathbf{p}} & \mathbb{O}_{2M_\star \times 2M_{\mathbf{u}}} \end{pmatrix}$$

Blocks must be squared

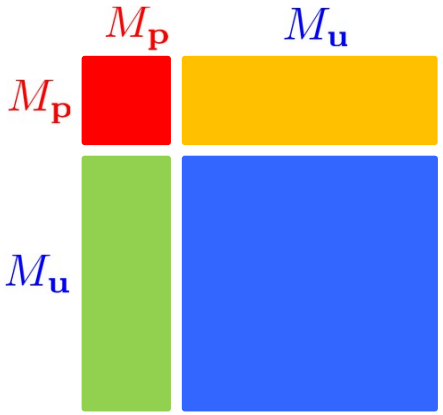
2D Neumann problem $\mathbf{p} = \bar{\mathbf{p}}$: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\,\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \bar{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$$

unknowns $\quad \boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p}}+2M_{\mathbf{u}}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$



$$\mathbb{E}^{(0)} = \begin{pmatrix} \mathbb{E}^{(0)}_{\mathcal{V}} & -\mathbb{E}^{(0)}_{\mathcal{K}} - \frac{1}{2}\mathbb{M}_{\mathbf{pu}} \\ \mathbb{M}_{\mathbf{up}} & \mathbb{O}_{2M_{\mathbf{u}} \times 2M_{\mathbf{u}}} \end{pmatrix}$$

$$\mathbb{E}^{(\ell)} = \begin{pmatrix} \mathbb{E}^{(\ell)}_{\mathcal{V}} & -\mathbb{E}^{(\ell)}_{\mathcal{K}} \\ \mathbb{O}_{2M_{\mathbf{u}} \times 2M_{\mathbf{p}}} & \mathbb{O}_{2M_{\mathbf{u}} \times 2M_{\mathbf{u}}} \end{pmatrix}$$

Blocks must be squared

2D Neumann problem $\mathbf{p} = \bar{\mathbf{p}}$: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\,\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \bar{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$$

unknowns $\quad \boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p}}+2M_{\mathbf{u}}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$

Displacement $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.0788	-	0.0745	-
0.1	0.0249	1.6627	0.0250	1.5754
0.05	0.0076	1.7049	0.0074	1.7457
0.025	0.0030	1.3268	0.0030	1.3231

Traction $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.1662	-	0.2409	-
0.1	0.0776	1.0982	0.0934	1.3676
0.05	0.0445	0.8021	0.0531	0.8130
0.025	0.0234	0.9282	0.0266	0.9990

Traction projected onto a space of functions continuous at the corner ?

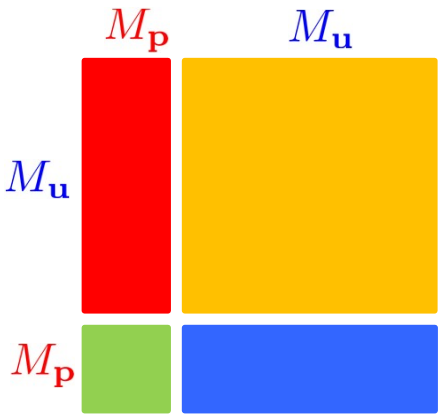
2D Neumann problem $\mathbf{p} = \bar{\mathbf{p}}$: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{0}_{\Delta x,d}\right]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \bar{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

unknowns $\quad \boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p}}+2M_{\mathbf{u}}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$



$$\mathbb{E}^{(0)} = \begin{pmatrix} \mathbb{E}^{(0)}_{\mathcal{V}} & -\mathbb{E}^{(0)}_{\mathcal{K}} - \frac{1}{2}\mathbb{M}_{\mathbf{uu}} \\ \mathbb{M}_{\mathbf{pp}} & \mathbb{O}_{2M_{\mathbf{p}} \times 2M_{\mathbf{u}}} \end{pmatrix}$$

$$\mathbb{E}^{(\ell)} = \begin{pmatrix} \mathbb{E}^{(\ell)}_{\mathcal{V}} & -\mathbb{E}^{(\ell)}_{\mathcal{K}} \\ \mathbb{O}_{2M_{\mathbf{p}} \times 2M_{\mathbf{p}}} & \mathbb{O}_{2M_{\mathbf{p}} \times 2M_{\mathbf{u}}} \end{pmatrix}$$

Blocks must be squared

2D Neumann problem $\mathbf{p} = \bar{\mathbf{p}}$: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{0}_{\Delta x,d}\right]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \overline{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

unknowns $\boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p}}+2M_{\mathbf{u}}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$

Displacement $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.0850	-	0.0800	-
0.1	0.0252	1.7523	0.0254	1.6568
0.05	0.0076	1.7330	0.0075	1.7664
0.025	0.0030	1.3215	0.0030	1.3283

Traction (10^{-16})

Same results considering $X^{-1}_{\Delta x,d}$

2D Neumann problem $\mathbf{p} = \bar{\mathbf{p}}$: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{0}_{\Delta x,d}\right]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = \left\langle \overline{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

unknowns $\boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p}}+2M_{\mathbf{u}}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$

Displacement $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.0850	-	0.0800	-
0.1	0.0252	1.7523	0.0254	1.6568
0.05	0.0076	1.7330	0.0075	1.7664
0.025	0.0030	1.3215	0.0030	1.3283

Traction (10^{-16})

Same results considering $X^{-1}_{\Delta x,d}$

Displacement $d = \bar{d} = 0$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.2713	-	0.2995	-
0.1	0.1740	0.6411	0.1909	0.0698
0.05	0.1035	0.7490	0.1155	0.7247
0.025	0.0619	0.7490	0.0698	0.7267

Traction (10^{-16})

Common to both approaches

2D Mixed problem: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

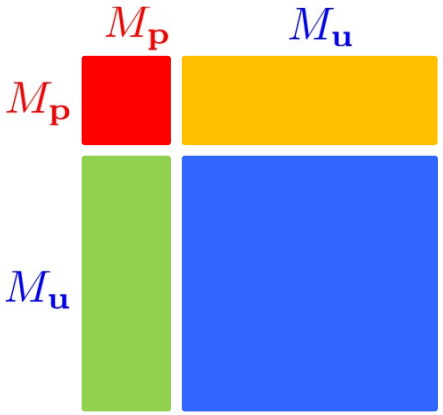
$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K} + \dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

$$\left\langle \dot{u}_i(\mathbf{x},t), \psi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_D)} = \left\langle \dot{\bar{u}}_i(\mathbf{x},t), \psi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_D)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_N)} = \left\langle \bar{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_N)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$$

unknowns

$$\boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p},D}+2M_{\mathbf{p},N}+2M_{\mathbf{u},D}+2M_{\mathbf{u},N}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$$



$$\mathbb{E}^{(0)} = \begin{pmatrix} \mathbb{E}^{(0)}_{\mathcal{V}} & -\mathbb{E}^{(0)}_{\mathcal{K}} - \frac{1}{2}\mathbb{M}_{\mathbf{pu}} \\ \mathbb{M}_{\mathbf{up}} & \frac{1}{2}\mathbb{M}_{\mathbf{uu}} \end{pmatrix}$$

$$\mathbb{E}^{(\ell)} = \begin{pmatrix} \mathbb{E}^{(\ell)}_{\mathcal{V}} & -\mathbb{E}^{(\ell)}_{\mathcal{K}} \\ \mathbb{O}_{2M_{\mathbf{u}} \times 2M_{\mathbf{p}}} & \mathbb{M}_{\mathbf{uu}} \end{pmatrix}$$

Blocks must be squared

2D Mixed problem: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

$$\left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} - \left\langle \left((\mathcal{K}+\dot{\mathcal{I}}/2)\,\mathbf{u}\right)_i(\mathbf{x},t), \phi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma)} = 0 \quad \forall \boldsymbol{\phi} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$$

$$\left\langle \dot{u}_i(\mathbf{x},t), \psi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_D)} = \left\langle \dot{\overline{u}}_i(\mathbf{x},t), \psi_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_D)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$$

$$\left\langle p_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_N)} = \left\langle \overline{p}_i(\mathbf{x},t), \dot{\psi}_i(\mathbf{x},t) \right\rangle_{L_2(\Sigma_N)} \quad \forall \boldsymbol{\psi} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$$

unknowns $\boldsymbol{\alpha}^{(\ell)} = \begin{pmatrix} \boldsymbol{\alpha}^{\mathbf{p}(\ell)} \\ \boldsymbol{\alpha}^{\mathbf{u}(\ell)} \end{pmatrix} \in \mathbb{R}^{2M_{\mathbf{p},D}+2M_{\mathbf{p},N}+2M_{\mathbf{u},D}+2M_{\mathbf{u},N}} \qquad \ell = 0, \ldots, N_{\Delta t-1}$

Displacement $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.0676	-	0.1673	-
0.1	0.0213	1.6696	0.0400	2.0648
0.05	0.0081	1.3963	0.0126	1.6612
0.025	0.0030	1.4509	0.0041	1.6235

Traction $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.2083	-	0.6290	-
0.1	0.0996	1.0649	0.2606	1.2711
0.05	0.0571	0.8029	0.1701	0.6153
0.025	0.0338	0.7542	0.0949	0.8420

2D Mixed problem: direct formulation, weak imposition of datum

Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

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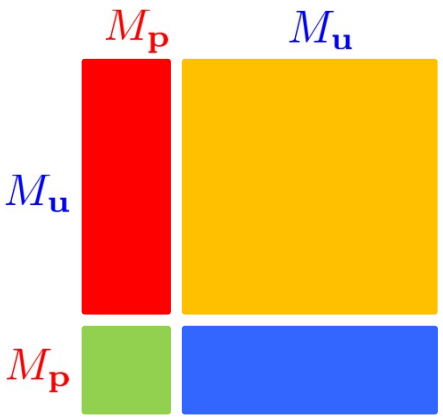

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Find $\mathbf{u} \in \left[V_{\Delta t,1} \times X^{0}_{\Delta x,d}\right]^2$ and $\mathbf{p} \in \left[V_{\Delta t,0} \times X^{-1}_{\Delta x,\bar{d}}\right]^2$ such that

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Displacement $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.0542	-	0.1223	-
0.1	0.0185	1.5505	0.0333	1.8783
0.05	0.0070	1.3984	0.0104	1.6839
0.025	0.0026	1.4548	0.0035	1.5518

Traction $d = \bar{d} = 1$

h	E_1	$rate_1$	E_2	$rate_2$
0.2	0.1195	-	0.3623	-
0.1	0.0662	0.8521	0.1989	0.8650
0.05	0.0378	0.8084	0.1215	0.7113
0.025	0.0238	0.6661	0.0751	0.6933

Energetic BEM for the Numerical Solution of 2D Elastodynamics Interior Problems, in preparation

Computation of matrix entries

Implementation of EBEM is not straightforward:
the computation of linear system entries depends on the problem at hand and on the used integral formulation

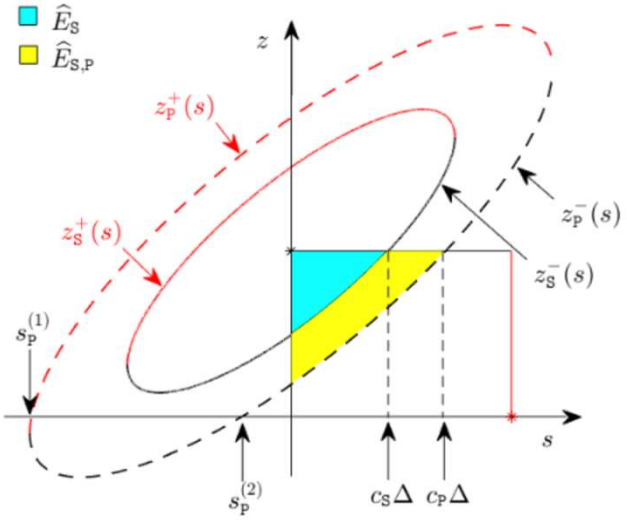
$$\frac{1}{2}\left\langle \dot{u}_i(\mathbf{x},t),\phi_i(\mathbf{x},t)\right\rangle_{L_2(\Sigma)} = \left\langle (\dot{\mathcal{V}}\mathbf{p})_i(\mathbf{x},t),\phi_i(\mathbf{x},t)\right\rangle_{L_2(\Sigma)} - \left\langle (\dot{\mathcal{K}}\mathbf{u})_i(\mathbf{x},t),\phi_i(\mathbf{x},t)\right\rangle_{L_2(\Sigma)}$$

$$\int_\Gamma \int_\Gamma w^{\mathbf{P}}_{\widetilde{m}}(\mathrm{x}) w^{\mathbf{P}}_{m}(\xi) \nu^{\mathcal{V}}_{ij}(\mathrm{r},\Delta) d\xi d\mathrm{x} \qquad i,j=1,2$$

2D

$$\begin{aligned}
\nu^{\mathcal{V}}_{ij}(\mathrm{r},\Delta) &= \left(\frac{r_i r_j}{r^4} - \frac{\delta_{ij}}{2r^2}\right)\left[\frac{H[c_\mathrm{p}\Delta - r]}{c_\mathrm{p}}\Delta\varphi_\mathrm{p}(r,\Delta) - \frac{H[c_\mathrm{s}\Delta - r]}{c_\mathrm{s}}\Delta\varphi_\mathrm{s}(r,\Delta)\right] \\
&+ \frac{\delta_{ij}}{2}\left[\frac{H[c_\mathrm{p}\Delta - r]}{c_\mathrm{p}^2}\hat{\varphi}_\mathrm{p}(r,\Delta) + \frac{H[c_\mathrm{s}\Delta - r]}{c_\mathrm{s}^2}\hat{\varphi}_\mathrm{s}(r,\Delta)\right]
\end{aligned}$$

$$\begin{aligned}
\varphi_\gamma &= \sqrt{c_\gamma^2\Delta^2 - r^2} \\
\hat{\varphi}_\gamma &= \log\left(\sqrt{c_\gamma^2\Delta^2 - r^2} + c_\gamma\Delta\right) - \log(r)
\end{aligned}$$



Double numerical quadrature

Highly accurate quadrature schemes for singular integrals in energetic BEM applied to elastodynamics, JCAM, (2022)

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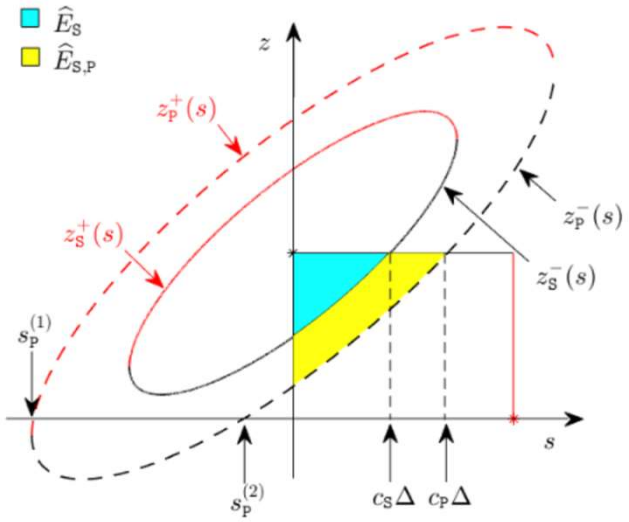
$$\int_\Gamma \int_\Gamma w^{\mathbf{p}}_{\widetilde{m}}(\mathrm{x}) w^{\mathbf{u}}_{m}(\xi) \nu^{\mathcal{K}}_{ij}(\mathrm{r},\Delta) d\xi d\mathrm{x} \qquad i,j=1,2$$

2D

$$\nu^{\mathcal{K}}_{ij}(\mathrm{r},\Delta) = \Big(\cdots\Big)_{\mathrm{p}} \frac{H[c_{\mathrm{p}}\Delta - r]}{c_{\mathrm{p}}} \frac{\varphi_{\mathrm{p}}(r,\Delta)}{r} - \Big(\cdots\Big)_{\mathrm{s}} \frac{H[c_{\mathrm{s}}\Delta - r]}{c_{\mathrm{s}}} \frac{\varphi_{\mathrm{s}}(r,\Delta)}{r}$$



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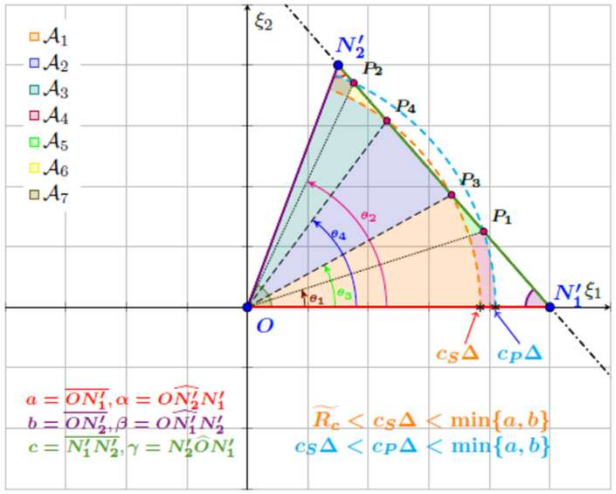
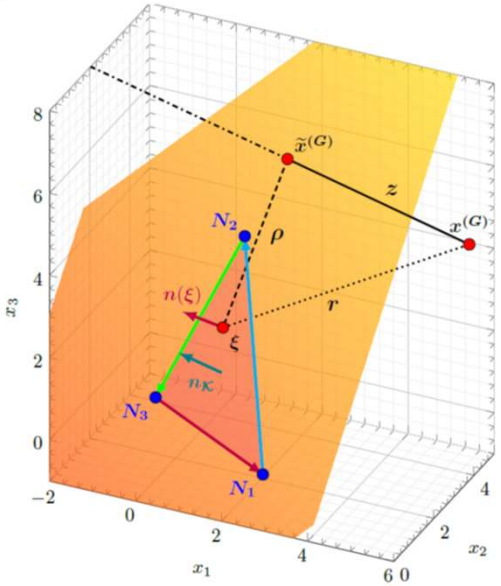
$$\int_\Gamma\int_\Gamma w^{\mathbf{P}}_{\widetilde{m}}(\mathrm{x})w^{\mathbf{P}}_{m}(\xi)\nu^{\mathcal{V}}_{ij}(\mathrm{r},\Delta)d\xi d\mathrm{x} \qquad i,j=1,2,3$$

3D

$$\begin{aligned}\nu^{\mathcal{V}}_{ij}(\mathrm{r},\Delta) &= \frac{r_i r_j}{r^3}\left[\frac{H[c_\mathrm{p}\Delta - r]}{c_\mathrm{p}^2} - \frac{H[c_\mathrm{s}\Delta - r]}{c_\mathrm{s}^2}\right] + \frac{\delta_{ij}}{r}\frac{H[c_\mathrm{s}\Delta - r]}{c_\mathrm{s}^2} \\ &- \frac{1}{2}\left(\frac{\delta_{ij}}{r^3} - 3\frac{r_i r_j}{r^5}\right)\left[\frac{H[c_\mathrm{p}\Delta - r]}{c_\mathrm{p}^2}\varphi_\mathrm{p}(r,\Delta) - \frac{H[c_\mathrm{s}\Delta - r]}{c_\mathrm{s}^2}\varphi_\mathrm{s}(r,\Delta)\right]\end{aligned}$$

$$\varphi_\gamma = c_\gamma^2\Delta^2 - r^2$$





Exterior numerical quadrature
Interior analytical integration

Single Layer approach in Energetic BEM approximation of 3D Elastodynamics, in preparation

2D

- Addition of hypersingular operator
- BEM-FEM coupling

3D

- Analysis of strongly singular operator

2D

- Addition of hypersingular operator
- BEM-FEM coupling

3D

- Analysis of strongly singular operator

Thank you for the attention and

Honor to Prof. E. Stephan!!!!

A. Aimi*, L. Desiderio*, G. Di Credico§, C. Guardasoni*