Boundary Elements and Friends
Innsbruck, Austria, August 25-27, 2022

Fast E-BEM by ACA compression for the resolution of acoustic and elastic 2D exterior problems in time domain

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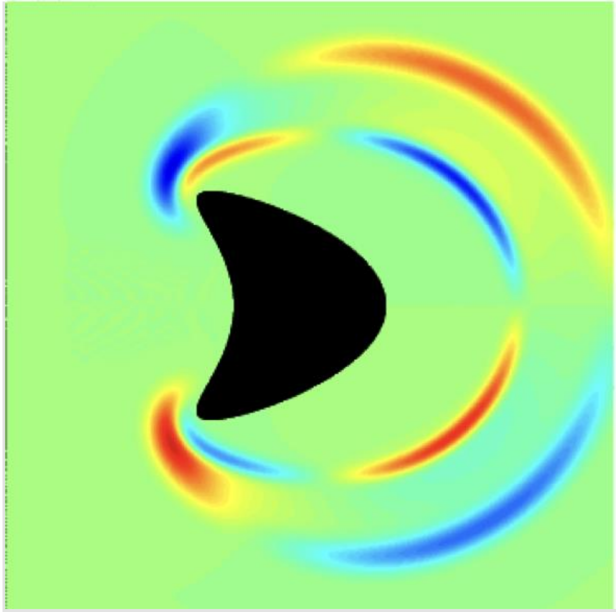

Joint work with:

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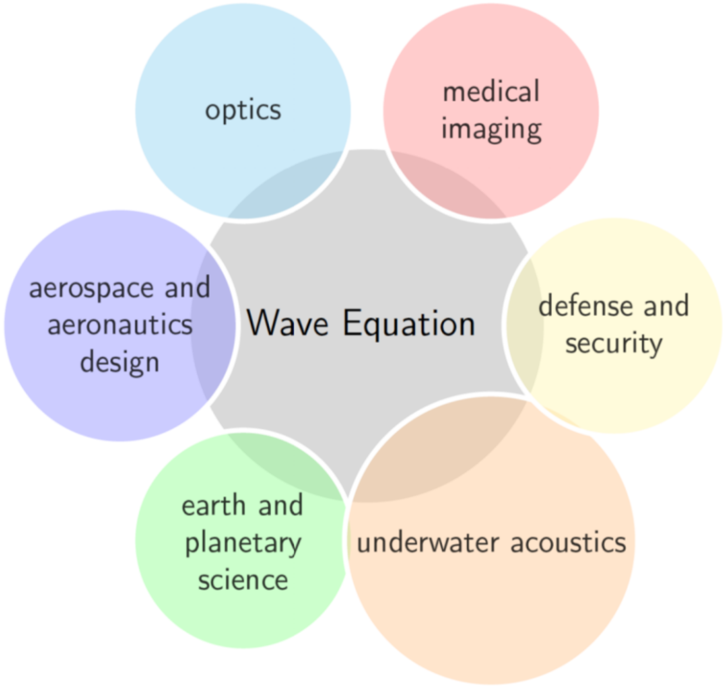
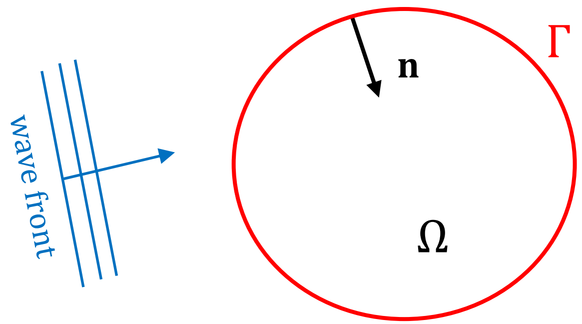


1. Acoustics and Elastic model problems (soft scattering)
2. Energetic Boundary Element Method (E-BEM)
3. ACA based E-BEM ➡ reduction of computational costs and memory requirements
4. Numerical results
5. Ongoing research

- Wave propagation into a linear, homogeneous and isotropic unbounded domain $\Omega_e \subset \mathbb{R}^2$:



$$\Omega_e = \mathbb{R}^2 \setminus \overline{\Omega}$$

Scattering by
a closed domain Ω

Acoustic Parameters
ρ mass density
c wave propagation speed

Elastic Parameters
ρ mass density
λ Lamé parameter
μ shear modulus

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$c_S = \sqrt{\frac{\mu}{\rho}}$$

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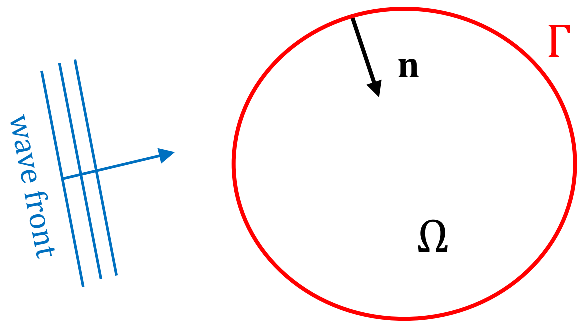

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$$\rho \ddot{\mathbf{u}}(\mathbf{x}, t) - \nabla \cdot \sigma[\mathbf{u}](\mathbf{x}, t) = 0, \qquad (\mathbf{x}, t) \in \Omega_e \times (0, T]$$

Vectorial Cauchy Equation of motion

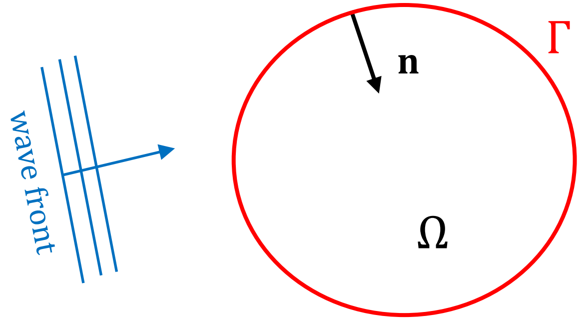
Acoustics: $\sigma[\mathbf{u}] = \rho c^2 \nabla \mathbf{u}$
Elastodynamics: $\sigma[\mathbf{u}] = \lambda(\nabla \cdot \mathbf{u})I + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$

$\mathbf{u}(\mathbf{x}, t) = (u_1, u_2)^\top(\mathbf{x}, t)$ → acoustic/elastic unknown field

$\mathbf{u}(\mathbf{x}, 0) = \dot{\mathbf{u}}(\mathbf{x}, 0) = 0$ for $\mathbf{x} \in \Omega_e$ and $\mathbf{u}(\mathbf{x}, t) = \mathbf{g}_{\mathcal{D}}(\mathbf{x}, t)$ for $(\mathbf{x}, t) \in \Gamma \times (0, T]$

1. ACOUSTIC and ELASTIC MODEL PROBLEMS (SOFT SCATTERING)

- Wave propagation into a linear, homogeneous and isotropic unbounded domain $\Omega_e \subset \mathbb{R}^2$:



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- We consider an indirect boundary integral representation formula for $\mathbf{u}$ in $\Omega_e \times (0, T]$:

$$\mathbf{u}(\mathbf{x}, t) = V\mathbf{\Phi}\,(\mathbf{x}, t) = \int_\Gamma \mathbf{G}^{\mathbf{uu}}(\mathbf{x} - \mathbf{y}; t) \star^{(t)} \mathbf{\Phi}(\mathbf{y}, t)\, d\Gamma_{\mathbf{y}}\,, \qquad (\mathbf{x}, t) \in \Omega_e \times (0, T]$$

single layer potential

$\mathbf{G}^{\mathbf{uu}}$ is the 2D fundamental solution

$\mathbf{\Phi}$ is an unknown density field belonging to the functional space of the traction vector $\mathbf{p} = (p_1, p_2)^\top$:

$$p_i(\mathbf{x}, t) = \sum_{h=1}^{2} \sigma_{ih}[\mathbf{u}](\mathbf{x}, t) n_h(\mathbf{x})\,, i = 1,2$$

1. ACOUSTIC and ELASTIC MODEL PROBLEMS (SOFT SCATTERING)

BOUNDARY INTEGRAL EQUATION

- The integral representation formula is strictly connected to the definition of the 2D fundamental solution $\mathbf{G}^{\mathbf{uu}} = \left(G_{ij}^{\mathbf{uu}}\right)_{i,j=1,2}$:

$$G_{ij}^{\mathbf{uu}}(\mathbf{x}-\boldsymbol{y};t-\tau) =$$

acoustics:

$$\delta_{ij}\frac{c}{2\pi}\frac{H[c(t-\tau)-r]}{\sqrt{c^2(t-\tau)^2-r^2}} \quad i,j=1,2$$

elastodynamics:

$$\frac{H[c_P(t-\tau)-r]}{2\pi\rho c_P}\left(\frac{r_ir_j}{r^4}\frac{2c_P^2(t-\tau)^2-r^2}{\sqrt{c_P^2(t-\tau)^2-r^2}}-\frac{\delta_{ij}}{r^2}\sqrt{c_P^2(t-\tau)^2-r^2}\right)$$
$$-\frac{H[c_S(t-\tau)-r]}{2\pi\rho c_S}\left(\frac{r_ir_j}{r^4}\frac{2c_S^2(t-\tau)^2-r^2}{\sqrt{c_S^2(t-\tau)^2-r^2}}-\frac{\delta_{ij}}{r^2}\frac{c_S^2(t-\tau)^2}{\sqrt{c_S^2(t-\tau)^2-r^2}}\right) \quad i,j=1,2$$

$$\mathbf{r} = \mathbf{x}-\boldsymbol{y}, \quad r = \|\mathbf{x}-\boldsymbol{y}\|$$

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$$\mathbf{r} = \mathbf{x} - \boldsymbol{y}, \quad r = ||\mathbf{x} - \boldsymbol{y}||$$

- Performing a limiting process of the spatial variable from the domain to the boundary, we get to the Boundary Integral Equation (BIE):

$$\mathbf{u}(\mathbf{x},t) = V\boldsymbol{\Phi}(\mathbf{x},t), \mathbf{x} \in \Omega_e, t \in (0,T] \quad \xrightarrow{\mathbf{x} \in \Omega_e \to \mathbf{x} \in \Gamma} \quad \mathbf{g}_{\mathcal{D}}(\mathbf{x},t) = V\boldsymbol{\Phi}(\mathbf{x},t), \mathbf{x} \in \Gamma, t \in (0,T]$$

representation formula → BIE

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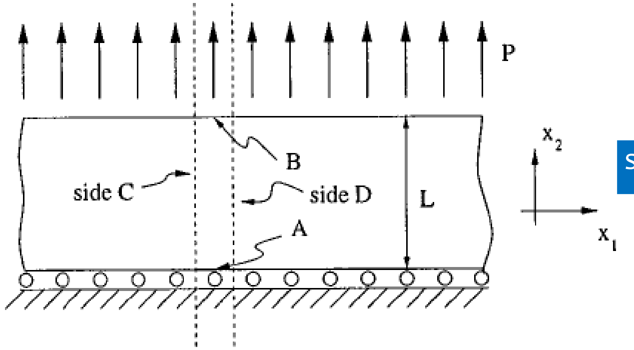
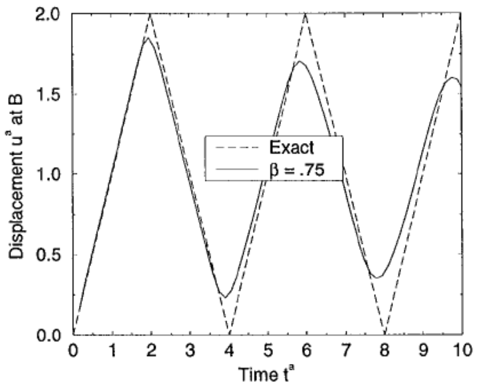
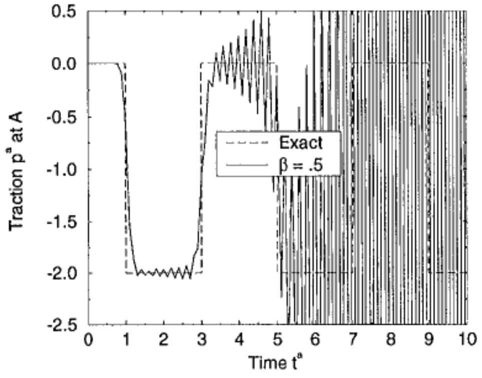
representation formula → BIE

- Standard discretization of the BIE may lead to unstable results for the approximation of the BIE unknown:



standard BEM





A. Frangi et al., On the numerical stability of time-domain elastodynamic analyses by BEM, CMAME, 1999

BOUNDARY INTEGRAL EQUATION and ITS ENERGETIC WEAK FORMULATION

- BIE in the unknown density $\mathbf{\Phi}$:

$$\mathrm{V}\mathbf{\Phi}\,(\mathbf{x},t) = \mathbf{g}_{\mathcal{D}}\,(\mathbf{x},t), \qquad \mathbf{x} \in \Gamma, t \in (0,T]$$

- Energetic Weak Formulation: find $\mathbf{\Phi} \in H^0\,[(0,T); H^{-1/2}(\Gamma)]$ solution of the energetic weak problem

$$\left\langle \frac{\partial}{\partial t}(\mathbf{V\Phi}), \boldsymbol{\phi} \right\rangle_{L^2(\Gamma\times(0,T])} = \left\langle \frac{\partial}{\partial t}\mathbf{g}_{\mathcal{D}}, \boldsymbol{\phi} \right\rangle_{L^2(\Gamma\times(0,T])} \qquad \forall \boldsymbol{\phi} \in H^0\,[(0,T); H^{-1/2}(\Gamma)]$$

$$:= B_{\mathcal{D}}(\mathbf{\Phi}, \boldsymbol{\phi})$$

$B_{\mathcal{D}}(\mathbf{\Phi}, \mathbf{\Phi}) = \mathcal{E}(\mathbf{u}, T)$ energy at the final instant T

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- The link with the energy $\mathcal{E}$ of the system allows to obtain good theoretical properties for the bilinear form $B_{\mathcal{D}}(\mathbf{\phi}, \boldsymbol{\phi})$.
- This energetic approach allows to overcome the instabilities arising from the standard weak form of the BIEs.

Acoustics: A. Aimi, M. Diligenti, C. Guardasoni, I. Mazzieri, S. Panizzi, IJNME (2009).
Elastodynamics: A. Aimi, L. Desiderio, C. Guardasoni, M. Diligenti, CAIM, (2019).

Energetic weak problem

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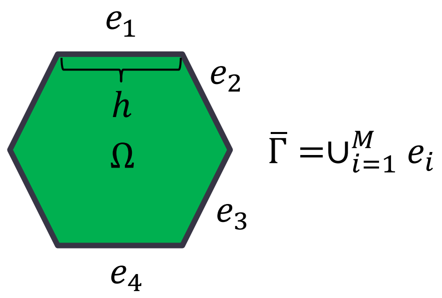
space-time approximation

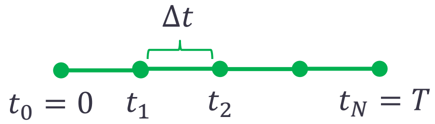
Discretized problem

$$\mathbb{E}_V \boldsymbol{a} = \boldsymbol{\beta}$$



$$\Phi_i(\mathbf{x}, t) \simeq \Phi_{i,h,\Delta t}(\boldsymbol{x}, t) = \sum_{k=0}^{N-1} \sum_{m=1}^{M} \alpha_{km}^{\mathbf{p},i} w_m^{\mathbf{p}}(\mathbf{x}) v_k(t)$$

piece-wise constants basis in space and time



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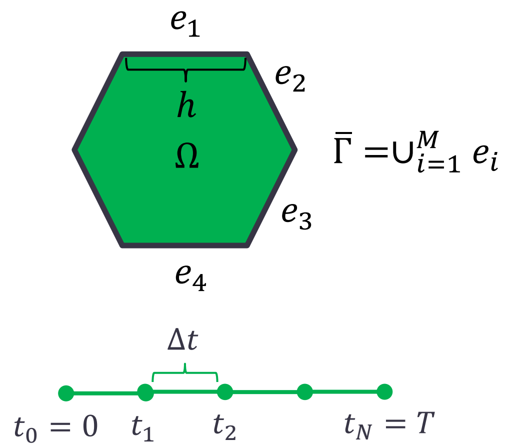

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piece-wise constants basis in space and time

Toeplitz block lower triangular matrix

$$\mathbb{E}_V = \begin{pmatrix} \mathbb{E}_V^{(0)} & 0 & 0 & 0 & 0 \\ \mathbb{E}_V^{(1)} & \mathbb{E}_V^{(0)} & 0 & 0 & 0 \\ \mathbb{E}_V^{(2)} & \mathbb{E}_V^{(1)} & \mathbb{E}_V^{(0)} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \mathbb{E}_V^{(N-1)} & \mathbb{E}_V^{(N-2)} & \mathbb{E}_V^{(N-3)} & \cdots & \mathbb{E}_V^{(0)} \end{pmatrix}$$

$$\mathbb{E}_V^{(l)} = \begin{pmatrix} \mathbb{E}_{V,11}^{(l)} & \mathbb{E}_{V,12}^{(l)} \\ \mathbb{E}_{V,21}^{(l)} & \mathbb{E}_{V,22}^{(l)} \end{pmatrix} \quad \forall l = 0, \dots, N-1$$

dimension $2M \times 2M$

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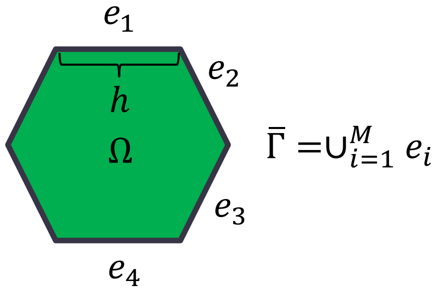

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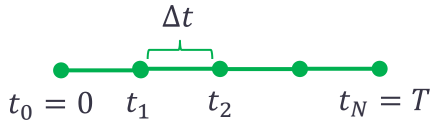

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$$\mathbb{E}_V^{(l)} = \begin{pmatrix} \mathbb{E}_{V,11}^{(l)} & \mathbb{E}_{V,12}^{(l)} \\ \mathbb{E}_{V,21}^{(l)} & \mathbb{E}_{V,22}^{(l)} \end{pmatrix} \quad \forall l = 0, \dots, N-1$$

dimension $2M \times 2M$

- Block forward substitution process to obtain the solution:

$$\mathbb{E}_V^{(0)} \boldsymbol{\alpha}_{(l)} = \boldsymbol{z}_{(l)} \qquad \forall l = 0, \dots, N$$

where

$$\boldsymbol{z}_{(l)} = \boldsymbol{\beta}_{(l)} - \sum_{j=1}^{l} \mathbb{E}_V^{(j)} \boldsymbol{\alpha}_{(l-j)},$$

✓ only the non-singular block $\mathbb{E}_V^{(0)}$ has to be inverted.

✓ for each l =1,…,N-1, the time blocks $\mathbb{E}_V^{(l)}$ are used to update the RHS at each time step

X for growing time, i.e. for growing index l, depending on wave speeds, blocks become fully populated – *$O(NM^2)$* overall cost

X for large M the updating of the RHS is expensive in terms of computations and memory requirements!

we need a compression technique to reduce the overall cost

ENTRIES of the ACOUSTICS TIME BLOCKS

$$\boxed{\mathbb{E}_V^{(l)}} = \begin{pmatrix} \mathbb{E}^{(l)} & 0 \\ 0 & \mathbb{E}^{(l)} \end{pmatrix} \qquad l = 0, \dots, N-1$$

structure of the acoustic time block

dimension $2M \times 2M$

- After a double analytic integration in the time variables, the entries of a generic acoustic block are expressed as follows:

$$\left(\mathbb{E}^{(l)}\right)_{\tilde{m},m} = -\sum_{\zeta,\xi=0}^{1} \frac{(-1)^{\xi+\zeta}}{2\pi} \int_\Gamma \int_\Gamma w_{\tilde{m}}^{\boldsymbol{p}}(\mathbf{x})\; w_m^{\mathbf{p}}(\mathbf{y})\; v(r;\Delta_{h+\xi,k+\zeta}) d\Gamma_{\mathbf{y}} d\Gamma_{\mathbf{x}}, \qquad \Delta_{h,k} = t_h - t_k$$

with

$$v(r;\Delta) = H[c\Delta - r]\left[\log\left(c\Delta + \sqrt{c^2\Delta^2 - r^2}\right) - \log(r)\right]$$

singularity of type $\mathcal{O}(\log(r))$ for $r \to 0$

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with

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- FUNDAMENTAL REMARK: if we set $\Delta_{hk} = t_l$, for growing time we have $ct_{l+1} > r$ and the block entries reduce to

$$\left(\mathbb{E}^{(l)}\right)_{\tilde{m},m} = -\frac{1}{2\pi} \int_\Gamma \int_\Gamma w_{\tilde{m}}^{\boldsymbol{p}}(\mathbf{x}) w_m^{\mathbf{p}}(\mathbf{y}) \tilde{\nu}(r; t_l) d\Gamma_{\mathbf{y}} d\Gamma_{\mathbf{x}}\,,$$

where the new compact kernel is

$$\tilde{\nu}(r,t) = \log\left(\frac{\left(ct + \sqrt{c^2 t^2 - r^2}\right)^2}{\left(c(t-\Delta t) + \sqrt{c^2(t-\Delta t)^2 - r^2}\right)\left(c(t+\Delta t) + \sqrt{c^2(t+\Delta t)^2 - r^2}\right)} \right)$$

and we do not observe any singularity since now the kernel is smooth for $r \longrightarrow 0$.
Moreover, $\tilde{\nu}(r,t) \longrightarrow 0$ for $t \longrightarrow +\infty$ and it is symmetric in space variables.

LOW-RANK REPRESENTATION of the TIME INTEGRATED ACOUSTICS KERNEL

- The reduction of the memory storage of the energetic BEM is related to the possibility of writing a low-rank representation or degenerate expansion of the kernel function, i.e.

$$\tilde{v}(\mathbf{x},\boldsymbol{y};t_l) = \sum_{k=0}^{k_l^*} \chi_k(\mathbf{x},t_l)\omega_k(\boldsymbol{y},t_l) + R_{k_l^*}(\mathbf{x},\boldsymbol{y},t_l)\,,$$

where $R_{k_l^*}(\mathbf{x},\mathbf{y};t_l)$ is the residuum that tends to zero for $k_l^* \to \infty$. It is well known that a kernel with the properties of $\tilde{v}$ possesses a low-rank representation in terms of eigenvalues and eigenfunctions in space variables ; however this is not useful to get to a closed form for this representation allowing to estimate the residuum with respect to a set rank k_l^*.

V. S. Vladimirov, Equations of Mathematical Physics, MIR, Moscow Russia, 1984.

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- Estimate of the low-rank: we consider a Taylor expansion in the $r = \|\, \mathbf{x} - \boldsymbol{y} \,\|$ variable, centered in $r = 0$:

$$\tilde{v}(r, t_l) = \sum_{k=0}^{k_l^*} C_k(t_l) r^{2k} + R_{k_l^*}(r, t_l) \coloneqq S_{k_l^*}(r, t_l) + R_{k_l^*}(r, t_l),$$

where, up to higher order terms

V. S. Vladimirov, Equations of Mathematical Physics, MIR, Moscow Russia, 1984.

LOW-RANK REPRESENTATION of the TIME INTEGRATED ACOUSTICS KERNEL

- The reduction of the memory storage of the energetic BEM is related to the possibility of writing a low-rank representation or degenerate expansion of the kernel function, i.e.

$$\tilde{v}(\mathbf{x}, \boldsymbol{y}; t_l) = \sum_{k=0}^{k_l^*} \chi_k(\mathbf{x}, t_l)\omega_k(\boldsymbol{y}, t_l) + R_{k_l^*}(\mathbf{x}, \boldsymbol{y}, t_l)\,,$$

where $R_{k_l^*}(\mathbf{x}, \mathbf{y}; t_l)$ is the residuum that tends to zero for $k_l^* \to \infty$. It is well known that a kernel with the properties of $\tilde{v}$ possesses a low-rank representation in terms of eigenvalues and eigenfunctions in space variables ; however this is not useful to get to a closed form for this representation allowing to estimate the residuum with respect to a set rank k_l^*.

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where, up to higher order terms

$$C_k(t_l) \cong (\Delta t c)^2 \frac{(2k+1)!}{(k!)^2 4^k} \frac{1}{(ct_l)^{2k+2}}.$$

Hence

$$S_{k_l^*}(r, t_l) \cong \tilde{S}_{k_l^*}(r, t_l) := \sum_{k=0}^{k_l^*} (\Delta t c)^2 \frac{(2k+1)!}{(k!)^2 4^k} \frac{r^{2k}}{(ct_l)^{2k+2}}$$

$$= \Delta t^2 c \frac{\partial^2}{\partial t_l \partial r} \left[\sum_{k=0}^{k_l^*} \frac{(2k)!}{(2k+1)!\,(k!)^2 4^k} \left(\frac{r}{ct_l} \right)^2 \right]$$

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$$= \Delta t^2 c \frac{\partial^2}{\partial t_l \partial r}\left[\sum_{k=0}^{k_l^*} \boxed{\frac{(2k)!}{(2k+1)!\,(k!)^2 4^k}\left(\frac{r}{ct_l}\right)^2}\right]$$

→ general term of the Taylor expansion of $F(x) = \mathrm{asin}(x)$ with $x = r/(ct_l)$

V. S. Vladimirov, Equations of Mathematical Physics, MIR, Moscow Russia, 1984.

LOW-RANK REPRESENTATION of ACOUSTICS BLOCKS

- Thus, for $k_l^* \to \infty$ we have:

$$\tilde{v}(r, t_l) \simeq \tilde{S}_\infty(r, t_l) = c\Delta t^2 \left(\frac{1}{t_l(c^2 t_l^2 - r^2)^{1/2}} + \frac{r^2}{t_l(c^2 t_l^2 - r^2)^{3/2}} \right)$$

We can conclude that it is possible to obtain a low-rank approximation of $\tilde{v}(r, t_l)$ and k_l^* is the rank required to achieve a given relative accuracy $\varepsilon > 0$ if

$$\left| R_{k_l^*}(r, t_l) \right| \simeq \left| \tilde{S}_\infty(r, t_l) - \tilde{S}_{k_l^*}(r, t_l) \right| \leq \varepsilon \, |\tilde{S}_\infty(r, t_l)|$$

LOW-RANK REPRESENTATION of ACOUSTICS BLOCKS

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- At the discrete level, the low rank representation of the kernel $\tilde{v}(r,t_l)$ implies that

$$\boxed{\mathbb{E}^{(l)} = \mathbb{S}_{k_l^*} + \mathbb{R}_{k_l^*} \quad \text{with} \quad \mathbb{S}_{k_l^*} = \mathbb{Q} \cdot \mathbb{W}^\top}$$

where $\mathbb{Q}$ and $\mathbb{W}$ are both $M \times k_l^*$ matrices and the residuum $\mathbb{R}_{k_l^*}$ is such that:

$$\left\| \mathbb{R}_{k_l^*} \right\|_F = \left\| \mathbb{E}^{(l)} - \mathbb{S}_{k_l^*} \right\|_F = \left\| \mathbb{E}^{(l)} - \mathbb{Q} \cdot \mathbb{W}^\top \right\|_F \le \varepsilon \left\| \mathbb{E}^{(l)} \right\|_F$$

LOW-RANK REPRESENTATION of ACOUSTICS BLOCKS

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- Remark. For $k_l^* \ll M$ we obtain a drastic reduction of the memory requirement for the storage of $\mathbb{E}^{(l)}$.

How to compute the matrices $\mathbb{Q}$ and $\mathbb{W}$?

ACA ALGORITHM FOR THE LOW-RANK APPROXIMATIONS OF ACOUSTICS BLOCKS

- Truncated SVD:

$$\mathbb{E}^{(l)} \simeq \sum_{k=0}^{k_l^*} \mathbf{q}_k \sigma_k \mathbf{w}_k^\top = \mathbb{Q}\Sigma\mathbb{W}^\top$$

✓ gives the best low-rank approximation

X requires the evaluation of all entries

X cost of the algorithm: $\mathcal{O}(M^3)$

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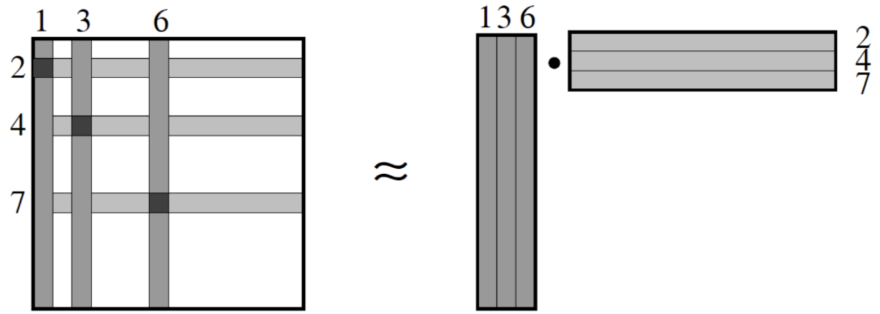
- Efficient algorithm to compute the low-rank approximations:

partially pivoted Adaptative Cross Approximation ; it basically consists in the computation of successive rank-1 approximations of $\mathbb{E}^{(l)}$:

$$\mathbb{E}^{(l)} = \mathbb{S}_{k_l^*} + \mathbb{R}_{k_l^*}, \qquad \mathbb{R}_{k_l^*} = \mathbb{E}^{(l)} - \sum_{k=1}^{k_l^*} \mathbf{q}_k \mathbf{w}_k^\top$$



✓ requires only few entries of the matrix
✓ in practice, $\mathcal{O}(k_l^* M)$ complexity

M. Bebendorf and S. Rjasanow, Computing, (2003)

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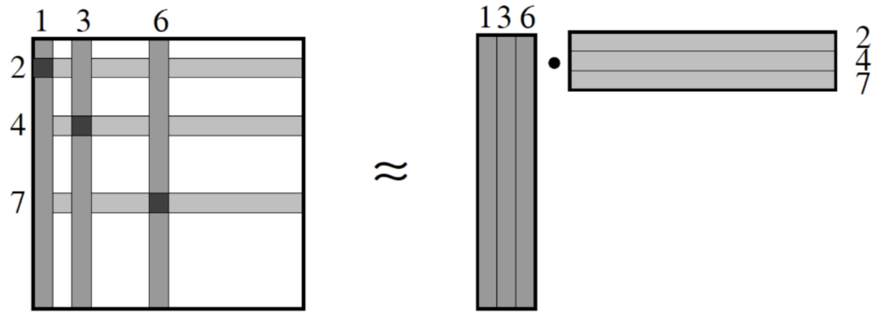


✓ requires only few entries of the matrix
✓ in practice, $\mathcal{O}(k_l^* M)$ complexity

M. Bebendorf and S. Rjasanow, Computing, (2003)

- Computational cost of the ACA based Energetic BEM: if $ct_{l^*} > \sqrt{3}diam(\Gamma)$, for any fixed $\varepsilon > 0$ and $l > l^*$, the rank of the block $\mathbb{E}^{(l)}$ can be bounded from above by the solution $k^*(\varepsilon)$ of the following inequality:

$$\frac{2(2k+3)!}{[(k+1)!]^2 12^{k+1}} < \varepsilon$$

➡ the ACA based compression scheme is bounded from the above by $2\ k^*(\varepsilon)(1-\lambda)NM$, where $\lambda = \frac{\sqrt{3}diam(\Gamma)}{cT}$

A. Aimi, L. Desiderio and G. Di Credico, CAMWA, (2022)

3. ACA based E-BEM

ENTRIES of the ELASTODYNAMICS TIME BLOCKS

$$\mathbb{E}_V^{(l)} = \begin{pmatrix} \mathbb{E}_{V,11}^{(l)} & \mathbb{E}_{V,12}^{(l)} \\ \mathbb{E}_{V,21}^{(l)} & \mathbb{E}_{V,22}^{(l)} \end{pmatrix} \qquad l = 0, \ldots, N-1$$

structure of the elastodynamic time block

dimension $2M \times 2M$

- After a double analytic integration in the time variables, the entries of a generic elastodynamic block are expressed as follows:

$$\left(\mathbb{E}^{(l)}\right)_{\tilde{m},m} = -\sum_{\zeta,\xi=0}^{1} \frac{(-1)^{\xi+\zeta}}{2\pi\rho} \int_\Gamma \int_\Gamma w_{\tilde{m}}^{\boldsymbol{p}}(\mathbf{x}) w_m^{\mathbf{p}}(\mathbf{y}) \nu\left(r; \Delta_{h+\xi,k+\zeta}\right) d\Gamma_{\mathbf{y}} d\Gamma_{\mathbf{x}}, \qquad \Delta_{h,k} = t_h - t_k$$

with

$$\nu_{ij}(r;\Delta) = \left(\frac{r_i r_j}{r^4} - \frac{\delta_{ij}}{2r^2}\right)\left[\frac{H[c_P\Delta - r]}{c_P}\varphi_P(r;\Delta) - \frac{H[c_S\Delta - r]}{c_S}\varphi_S(r;\Delta)\right] + \frac{\delta_{ij}}{2}\left[\frac{H[c_P\Delta - r]}{c_P}\hat{\varphi}_P(r;\Delta) + \frac{H[c_S\Delta - r]}{c_S}\hat{\varphi}_S(r;\Delta)\right]$$

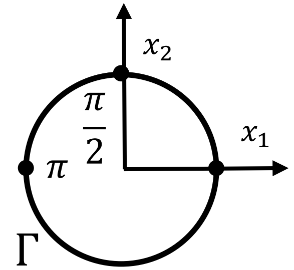
singularity of type $\mathcal{O}(\log(r))$ for $r \to 0$

having set

$$\varphi_\gamma(r;\Delta) = c_\gamma \Delta \sqrt{c_\gamma^2 \Delta^2 - r^2}, \qquad \hat{\varphi}_\gamma(r;\Delta) = \log\left(c_\gamma \Delta + \sqrt{c_\gamma^2 \Delta^2 - r^2}\right) - \log(r)$$

- Having set $\Delta_{hk} = t_l$, for growing time we have $ct_{l+1} > r$ and in the above sum singularity disappears.
- The previously applied block low-rank ACA approximation procedure can be applied also in the elastodynamics context

ACOUSTICS: EFFICIENCY and ACCURACY



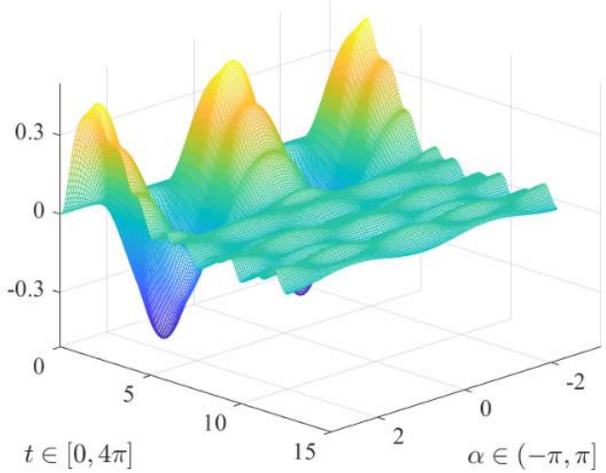
Boundary datum: $g_{\mathcal{D},1}(x,t) = t^4 e^{-t}\cos(x_1^2 + 2x_2^2)$ and $g_{\mathcal{D},2}(x,t) = 0$

Threshold: $\varepsilon_{ACA} = 1.0e-04$

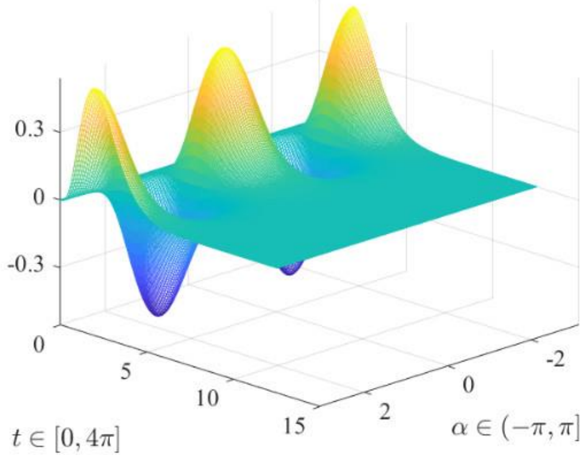
Absolute Error: $E_{L^2} := \max_{t\in(0,T]} \left\| \mathbf{\Phi}^*_{M_e,N_e}(\cdot,t) - \mathbf{\Phi}_{M,N}(\cdot,t) \right\|_{L^2(\Gamma)}$ $\quad (M_e = 4096, N_e = 8192, T = 4\pi)$

Memory Saving: $mem(\%) := 100 \cdot \left(1 - \frac{1}{N}\sum_{l=0}^{N-1} \frac{2k_l^*}{M}\right)$

		$C = 1$			$C = 343$		
M	N	E_{L^2}	EOC	$mem(\%)$	E_{L^2}	EOC	$mem(\%)$
8	16	$2.67e-01$		0.0%	$5.94e-01$		37.5%
			0.6			0.7	
16	32	$1.77e-01$		0.0%	$3.54e-01$		67.2%
			1.0			1.0	
32	64	$8.47e-02$		39.0%	$1.78e-01$		84.6%
			1.0			1.0	
64	128	$4.22e-02$		60.9%	$8.95e-02$		92.1%
			1.0			1.0	
128	256	$2.03e-02$		72.1%	$4.41e-02$		96.0%
			1.1			1.0	
256	512	$9.44e-03$		77.3%	$2.13e-02$		96.0%
			1.2			1.1	
512	1024	$4.04e-03$		79.7%	$9.93e-03$		98.3%



$\Phi^\star_{1,M_e,N_e}(\alpha,t)$, $C = 1$



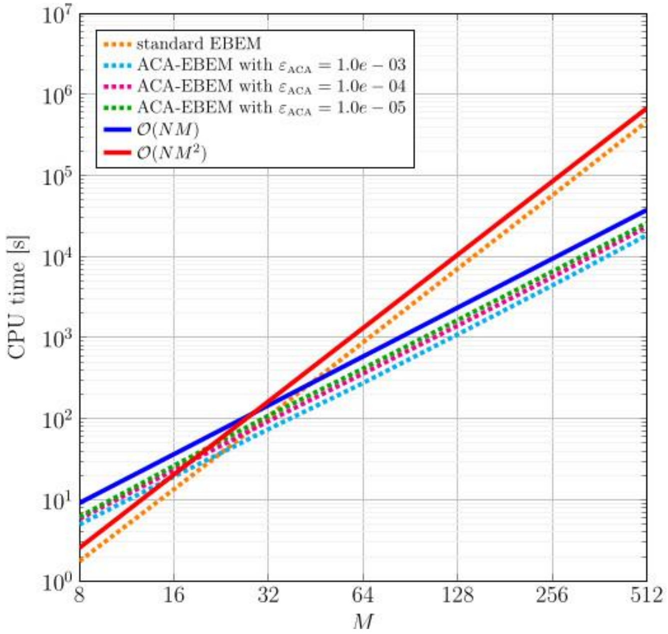
$\Phi^\star_{1,M_e,N_e}(\alpha,t)$, $C = 343$

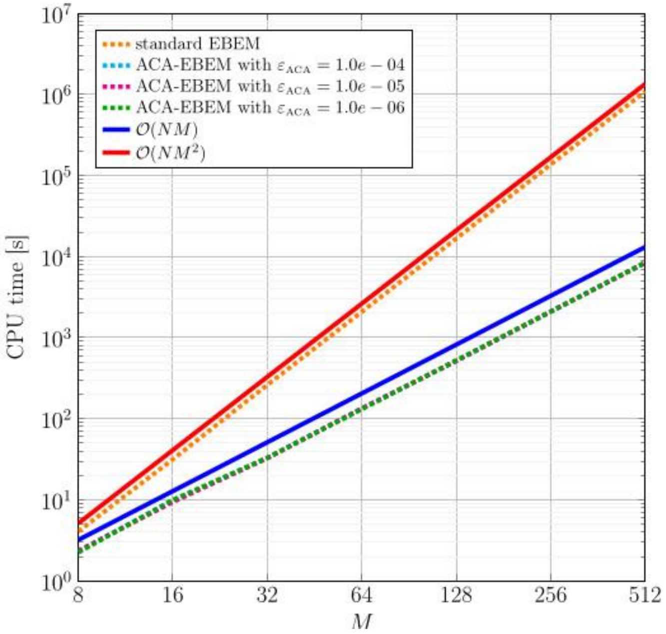
4. NUMERICAL RESULTS

ACOUSTICS: SAVING of CPU TIME and GLOBAL TIME HISTORY of the ERROR

CPU TIME

The growth of the CPU time (measured in seconds) with ACA compression is optimal, i.e. $\mathcal{O}(NM)$.



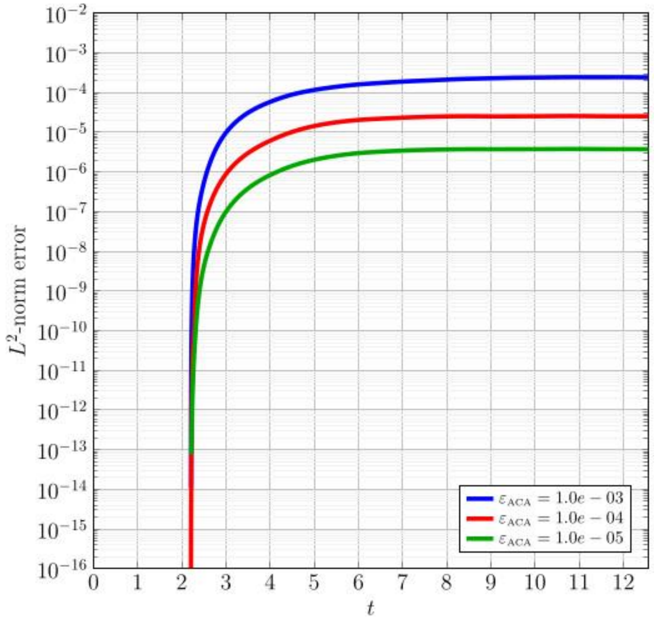


TIME HISTORY OF THE ERROR

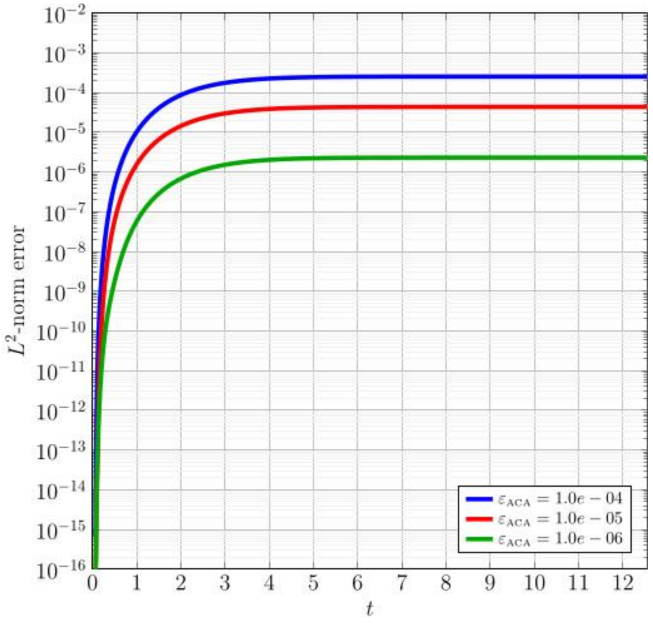
$$E_{L^2}(t) := \left\| \mathbf{\Phi}^*_{M_e,N_e}(\cdot,t) - \mathbf{\Phi}_{M,N}(\cdot,t) \right\|_{L^2(\Gamma)}$$

$$(M_e = M = 4096, N_e = N = 8192)$$

For growing time, the level of error introduced by the ACA can be controlled by the parameter ε_{ACA}, since the former is at most of the same order of magnitude of the latter.



$C = 1$



$C = 343$

4. NUMERICAL RESULTS

ACOUSTICS: APPLICATION to SCATTERING PROBLEMS

- Scattering of a packet of four plane waves impacting on a circumference of radius 1

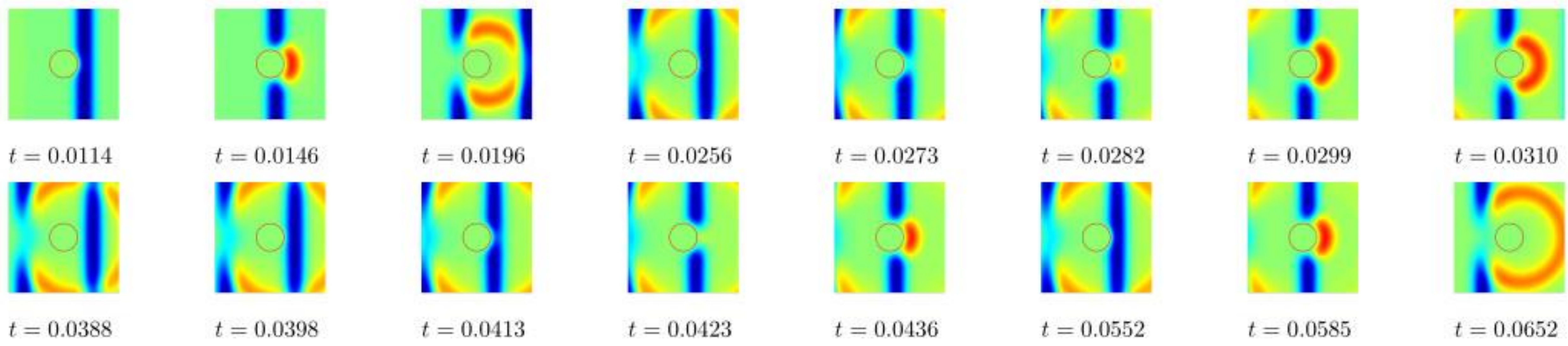
Boundary datum: $g_{\mathcal{D},1}(x,t) = -\sum_{i=1}^{4} e^{\left(-2\left(x_1-\xi_i+c(t+t_0)\right)^2\right)}$ and $g_{\mathcal{D},2}(x,t) = 0$;

Parameters: $\xi = (50,55,60,65)^{\mathsf{T}}, \mathrm{t}_0 = 0.13, c = 343$ and T =0.15

DOF: $M = 128$ and $N = 1049$,

Threshold: $\varepsilon_{ACA} = 1.0e-04 \Longrightarrow mem(\%) = 87.3$

- Representation of the horizontal component of the reconstructed total field in a region surrounding the scatterer



ACOUSTICS: APPLICATION to SCATTERING PROBLEMS

- Scattering of a plane wave impinging on a flower-shaped scatterer

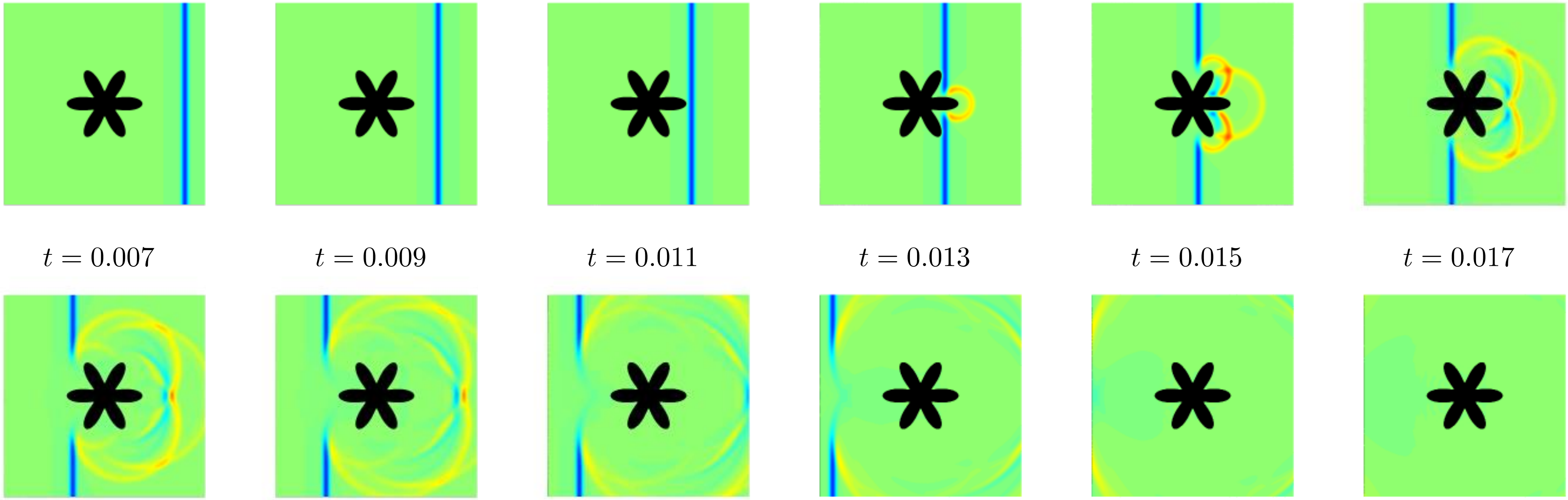
Boundary datum: $g_{\mathcal{D},1}(x,t) = e^{\left(-50\left(x_1-50+c(t+0.13)\right)^2\right)}$ and $g_{\mathcal{D},2}(x,t) = 0$;

Parameters: $c = 343$ and $T = 0.07$

DOF: $M = 4096$ and $N = 10352$

Threshold: $\varepsilon_{ACA} = 1.0e-04 \Longrightarrow mem(\%) = 86.2$

- Representation of the horizontal component of the reconstructed total field in a region surrounding the scatterer



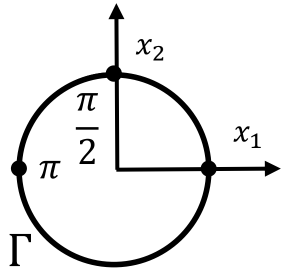
$t = 0.007$ $t = 0.009$ $t = 0.011$ $t = 0.013$ $t = 0.015$ $t = 0.017$

$t = 0.019$ $t = 0.021$ $t = 0.023$ $t = 0.025$ $t = 0.027$ $t = 0.029$

ELASTODYNAMICS: EFFICIENCY and ACCURACY

Boundary datum: $g_{\mathcal{D},1}(x,t) = t^4 e^{-t} x_1$ and $g_{\mathcal{D}2}(x,t) = t^4 e^{-t} x_2$;

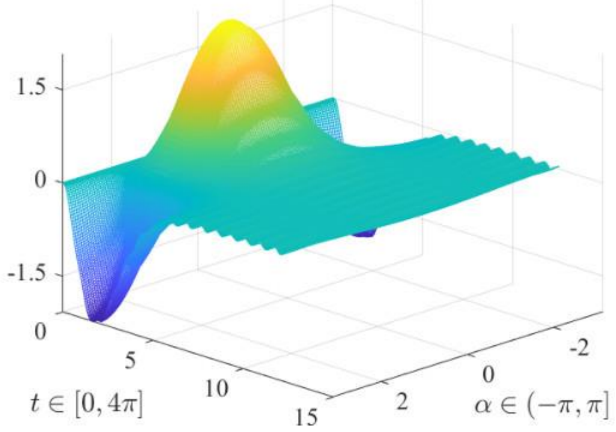
Threshold: $\varepsilon_{ACA} = 1.0e-04$



Absolute Error: $E_{L^2} := \max_{t \in (0,T]} \left\| \mathbf{\Phi}^*_{M_e,N_e}(\cdot,t) - \mathbf{\Phi}_{M,N}(\cdot,t) \right\|_{L^2(\Gamma)}$ $(M_e = 512, N_e = 2048, T = 4\pi)$

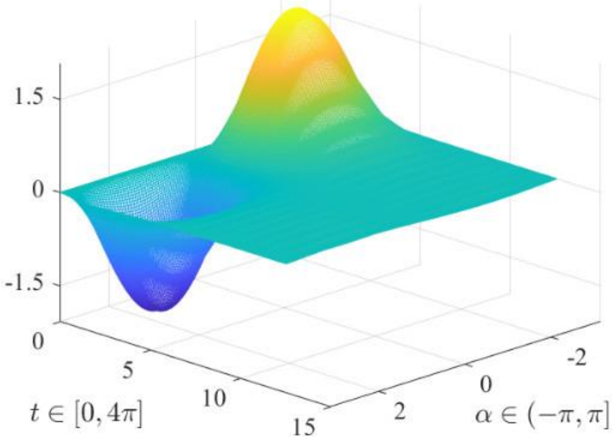
Memory Saving: $mem(\%) := 100 \cdot \left(1 - \frac{1}{N} \sum_{ij=1}^{2} \sum_{l=0}^{N-1} \frac{2k^*_{ij,l}}{M}\right)$

		$C_s = 1, C_P = 2$		
M	N	E_{L^2}	EOC	$mem(\%)$
8	32	6.29e−01		0.0%
			1.0	
16	64	3.11e−01		0.0%
			1.1	
32	128	1.49e−01		38.7%
			1.2	
64	256	6.44e−02		60.2%
			1.5	
128	512	2.15e−02		71.0%



$\Phi^\star_{1,M_e,N_e}(\alpha,t)$

$\Phi^\star_{2,M_e,N_e}(\alpha,t)$



ELASTODYNAMICS: APPLICATION to SCATTERING PROBLEMS

- Scattering of an incident P-wave impacting on a circumference of radius 1

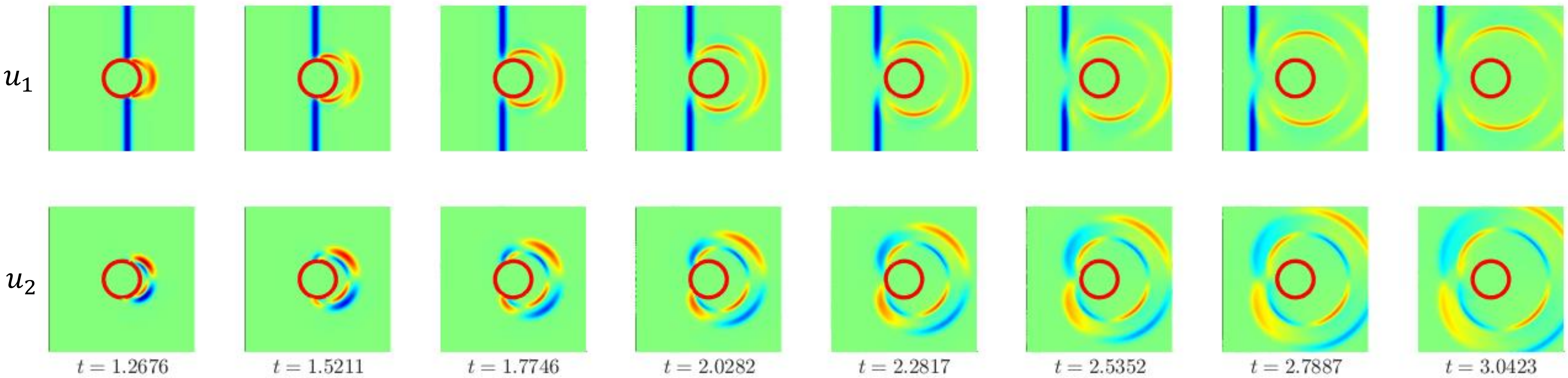
Boundary datum: $g_{\mathcal{D},1}(x,t) = e^{(-20(x_1-2+c_P t-0.475)^2)}$ and $g_{\mathcal{D},2}(x,t) = 0$

Parameters: $c_S = 1, c_P = \sqrt{3}$ and T =12

DOF: $M = 128$ and $N = 426$

Threshold: $\varepsilon_{ACA} = 1.0e-04 \Longrightarrow mem(\%) = 71.1$

- Representation of the horizontal and vertical components of the reconstructed total field in a region surrounding the scatterer



ELASTODYNAMICS: APPLICATION to SCATTERING PROBLEMS

- Scattering of an incident P-wave impacting on a kite-shaped scatterer

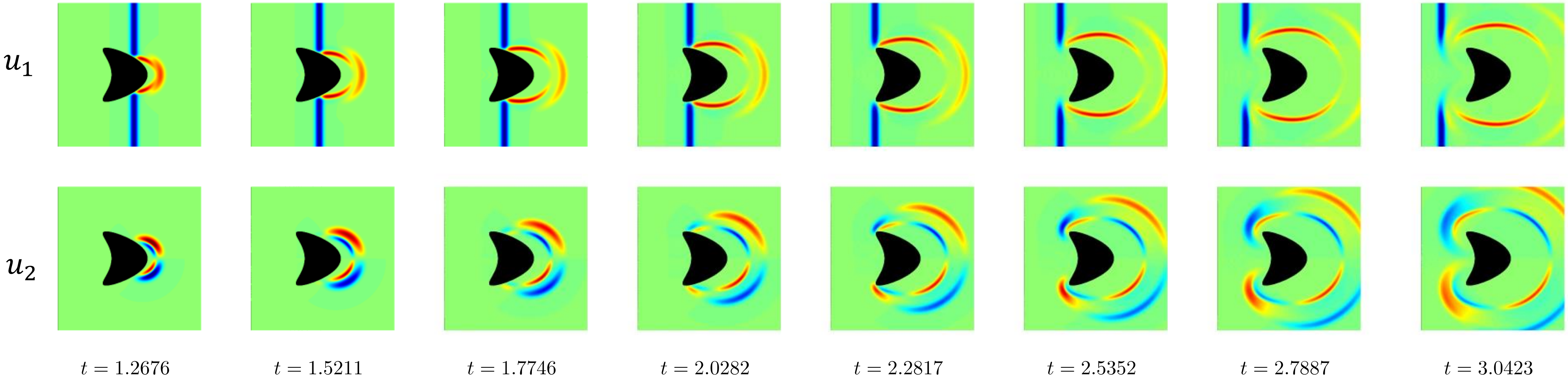
Boundary datum: $g_{\mathcal{D},1}(x,t) = e^{(-20(x_1-2+c_P t-0.475)^2)}$ and $g_{\mathcal{D},2}(x,t) = 0$;

Parameters: $c_S = 1, c_P = \sqrt{3}$ and $T = 12$

DOF: $M = 8192$ and $N = 28380$

Threshold: $\varepsilon_{ACA} = 1.0e-04 \Longrightarrow mem(\%) = 72.5$

- Representation of the horizontal and vertical components of the reconstructed total field in a region surrounding the scatterer



ACOUSTIC and ELASTIC HARD SCATTERING PROBLEMS

- PDE: $\rho\ddot{\mathbf{u}}(\mathbf{x},t) - \nabla\cdot\sigma[\mathbf{u}](\mathbf{x},t) = 0, \qquad (\mathbf{x},t)\in\Omega_e\times(0,T]$

$$\begin{cases}\mathbf{u}(\mathbf{x},0) = \dot{\mathbf{u}}(\mathbf{x},0) = 0, & \mathbf{x}\in\Omega_\mathrm{e}\\ \mathbf{p}(\mathbf{x},t) = \mathbf{g}_N(\mathbf{x},t), & (\mathbf{x},t)\in\Gamma\times(0,T]\end{cases}$$

- BIE: $\mathbf{g}_N(\mathbf{x},t) = \mathrm{W}\boldsymbol{\psi}\,(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma, t\in(0,T]$

where $\mathrm{W}\boldsymbol{\psi}(\mathbf{x},t) = \int_\Gamma \mathbf{G}^{\mathbf{pp}}(\mathbf{x}-\mathbf{y};t) \star^{(t)} \boldsymbol{\psi}(\mathbf{y},t)\,d\Gamma_\mathbf{y}\,,$ with $\mathbf{G}^{\mathbf{pp}}(\mathbf{x}-\mathbf{y};t) = \sigma_\mathbf{x}[\sigma_\mathbf{y}[\mathbf{G}^{\mathbf{uu}}]]$

- Energetic BEM: $\mathbb{E}_W\boldsymbol{a} = \boldsymbol{\beta}$ Toeplitz block lower triangular matrix

- After a double analytic integration in the time variables, the entries of a generic time block are double integrals over $\Gamma\times\Gamma$, whose kernel has a hyper−singularity of type $\mathcal{O}(1/r^2)$ for $r\to 0$

- For growing time, the kernel singularity disappears and time blocks are vanishing with decreasing rank.

- The previously applied block low-rank ACA approximation procedure can be applied also in the context of hard scattering.

ACOUSTICS AND ELASTODYNAMICS HARD SCATTERING : EFFICIENCY and ACCURACY

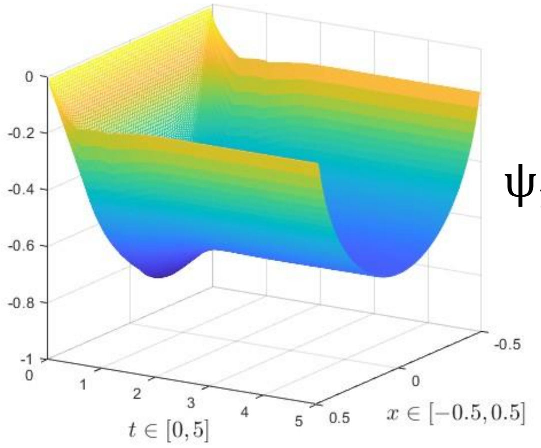
(Components of the) Boundary datum on a rectilinear scatterer of unitary length: $g_{\mathcal{N},1}(x,t) = g_{\mathcal{N},2}(x,t) = H[t]$;

Threshold: $\varepsilon_{ACA} = 1.0e-04$

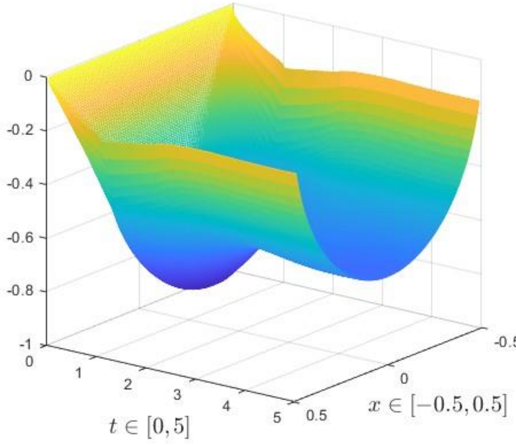
Absolute error evaluated in energy norm

Memory Saving: $mem(\%) := 100 \cdot \left(1 - \frac{1}{N}\sum_{l=0}^{N-1}\frac{2k_l^*}{M}\right)$

$T = 5$		*Acoustics* $(c_S = 1)$			*Elastodynamics* $(c_S = 1, c_P = 2)$		
M	N	E	EOC	$mem(\%)$	E	EOC	$mem(\%)$
9	50, 100	$1.93e-01$	0.5	0.0%	$2.27e-01$	0.5	0.0%
19	100, 200	$1.37e-01$	0.5	11.2%	$1.59e-01$	0.5	41.2%
39	200, 400	$9.59e-02$	0.5	31.2%	$1.12e-01$	0.5	59.7%
79	400, 800	$6.70e-02$	0.5	50.2%	$7.91e-02$	0.5	69.8%
159	800, 1600	$4.71e-02$		64.7%	$5.50e-02$		75.9%



$\psi_1(x,t)$ (elastic)



$\psi_2(x,t)$ (elastic)

A. Aimi, G. Di Credico, H. Gimperlein, E. P. Stephan, Numer. Math., (2021, under review)

- A. Aimi, M. Diligenti, C. Guardasoni, I. Mazzieri, S. Panizzi, An energy approach to space-time Galerkin BEM for wave propagation problems, *International Journal for Numerical Methods in Engineering*, 2009.

- A. Aimi, L. Desiderio, C. Guardasoni, M. Diligenti, Application of Energetic BEM to 2D Elastodynamic Soft Scattering Problems, *Communications in Applied and Industrial Mathematics*, 2019.

- A. Aimi, G. Di Credico, M. Diligenti, C. Guardasoni, Highly accurate quadrature schemes for singular integrals in energetic BEM applied to elastodynamics, *Journal of Computational and Applied Mathematics*, 2022.

- A. Aimi, G. Di Credico, H. Gimperlein, E. P. Stephan, Higher-order time domain boundary elements for elastodynamics - graded meshes and hp-versions, *Numerische Mathematik* , 2021 (under review).

- A. Aimi, L. Desiderio, G. Di Credico, Partially pivoted ACA based acceleration of the Energetic BEM for time-domain acoustic and elastic waves exterior problems, *Compuetrs & Mathematics with Applications*, 2022.

THANK YOU FOR THE ATTENTION AND…
HAPPY BIRTHDAY ERNST STEPHAN!☺