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# DYNAMICAL FRICTIONAL CONTACT BY SPACE-TIME BOUNDARY ELEMENTS


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# Motivation



- Contact problems play an important role in numerous **applications in mechanics**, from fracture dynamics and crash tests to rolling car tires.

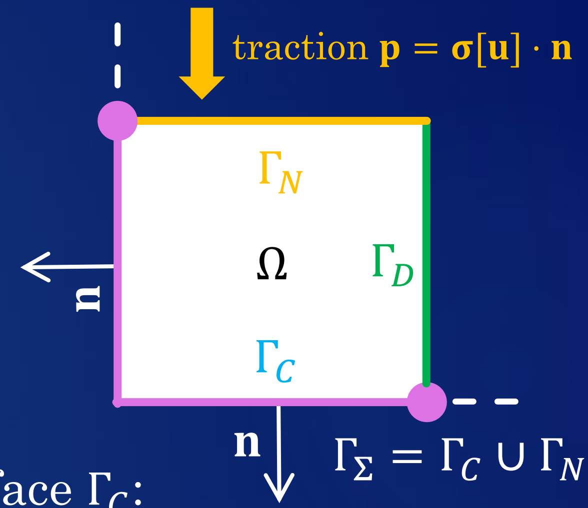
- From the mathematical point of view, a **set of non linear conditions** is imposed in order **to model the frictional contact** at the **interface between an elastic body and a rigid surface.**

- The mathematical formulation of contact problems can rely on **boundary integral operators**. This naturally induces a numerical solution approach based on the **Boundary Element Method.**

# Unilateral frictional contact problem

- $\sigma[\mathbf{u}] = \varrho(c_P^2 - c_S^2)(\nabla\cdot\mathbf{u})I + \varrho c_S^2(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$ (elastodynamic Cauchy stress tensor)

$$\begin{cases} \nabla\cdot\sigma[\mathbf{u}] = \varrho\ddot{\mathbf{u}}, & \mathbf{x}\in\Omega, t\in(0,T] \\ \mathbf{u} = \dot{\mathbf{u}} = \mathbf{0} & \mathbf{x}\in\Omega, t\leq 0 \\ \mathbf{u} = \mathbf{0}, & \mathbf{x}\in\Gamma_D, t\in(0,T] \\ \mathbf{p} = \mathbf{f}, & \mathbf{x}\in\Gamma_N, t\in(0,T] \end{cases}$$




- **Frictional contact conditions*** on the interface $\Gamma_C$:

$\|$: *tangential* conditions on $\Gamma_C$

$$\begin{cases} |p_\|| \leq \mathcal{F} \\ |p_\|| < \mathcal{F} \Longrightarrow \dot{u}_\| = 0 \\ |p_\|| = \mathcal{F} \Longrightarrow \exists\alpha\geq 0 : \dot{u}_\| = -\alpha p_\| \end{cases}$$

$\perp$*: orthogonal conditions on* $\Gamma_C$

$$\begin{cases} u_\perp \geq g, p_\perp \geq f_\perp \\ u_\perp > g \Longrightarrow p_\perp = f_\perp \end{cases}$$

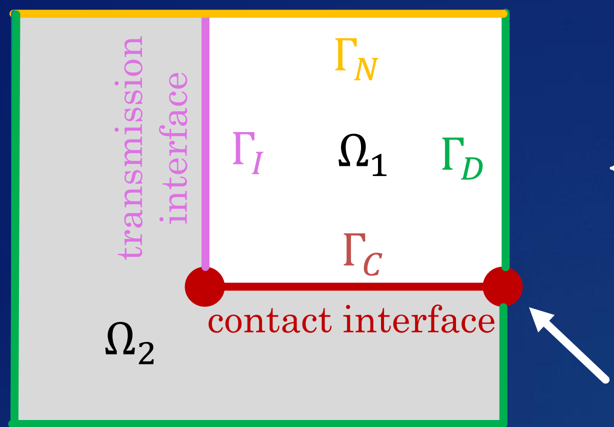
- $\boldsymbol{\mathcal{F}}$ **represents the friction threshold** (Tresca or Coulomb type)
- $u_\perp \geq g \Longrightarrow$ **the body cannot penetrate the contact interface**.

*adopted convention: $u_\perp = -\mathbf{u}\cdot\mathbf{n}$

# Two - body frictional contact problem



- Generalization to a **frictional contact problem between two linearly elastic bodies** $\Omega_1$ and $\Omega_2$ (different material properties*):




$$\begin{cases} \nabla \cdot \sigma_j[\mathbf{u}_j] = \varrho \ddot{\mathbf{u}}_j, & \mathbf{x} \in \Omega_j, t \in (0,T] \\ \mathbf{u}_j = \dot{\mathbf{u}}_j = \mathbf{0} & \mathbf{x} \in \Omega_j, t \leq 0 \\ \mathbf{u}_j = 0, & \mathbf{x} \in \Gamma_D, t \in (0,T] \\ \mathbf{p}_j = \mathbf{f}, & \mathbf{x} \in \Gamma_N, t \in (0,T] \end{cases}$$

- Transmission and contact conditions:

$$\begin{cases} \mathbf{u_1} - \mathbf{u_2} = \mathbf{g}, & \mathbf{x} \in \Gamma_I, t \in (0,T] \\ \mathbf{p_1} + \mathbf{p_2} = \mathbf{f}, & \mathbf{x} \in \Gamma_I, t \in (0,T] \end{cases}$$

$$\begin{cases} u_{1,\perp} - u_{2,\perp} \geq g_\perp, \quad p_{1,\perp} \geq 0, p_{2,\perp} \geq f_\perp, \quad \begin{cases} p_{1,\perp} = f_\perp - p_{2,\perp} = 0 & \text{if} \quad u_{1,\perp} - u_{2,\perp} > g_\perp \\ p_{1,\perp} + p_{2,\perp} = f_\perp & \text{if} \quad u_{1,\perp} - u_{2,\perp} = g_\perp \end{cases} \\ p_{1,\parallel} + p_{2,\parallel} - f_\parallel = 0, \quad |p_{1,\parallel}| < \mathcal{F}, \quad p_{1,\parallel}(\dot{u}_{1,\parallel} - \dot{u}_{2,\parallel} - \dot{g}_\parallel) + \mathcal{F}\,|\dot{u}_{1,\parallel} - \dot{u}_{2,\parallel} - \dot{g}_\parallel| = 0 \end{cases}$$

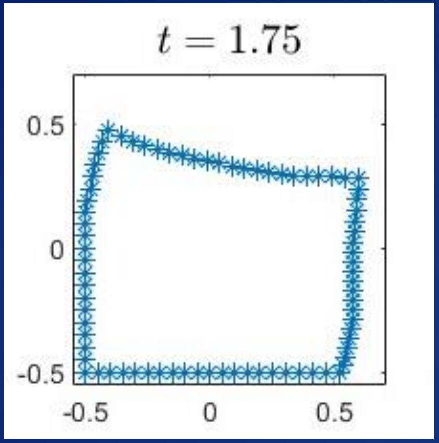
- Small relative deformations: two bodies share a **non-penetrable contact interface $\Gamma_C$**, allowing the opening of a small gap, while **the transmission interface $\Gamma_I$ takes into account a rigid connection** between $\Omega_1$ and $\Omega_2$.

*$\sigma_j$ constructed w.r.t the outward normal of $\Omega_j$ for $j = 1,2$

# Different types of friction laws

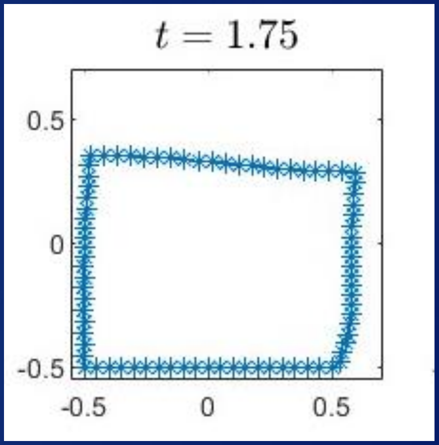
- The adopted friction law is specified by the friction threshold $\mathcal{F} \geq 0$:

**Tresca friction**:
prescribed threshold $\mathcal{F} \in L^\infty(\Gamma_C)$,
independent of the traction $\mathbf{p}$




- ✓ It allows theoretical analysis for a priori error estimate
- X it leads to unphysical tangential forces when the gap between the body and the contact interface is not trivial

**Coulomb Friction**:
threshold $\mathcal{F}$ is a linear function of the traction:
$\mathcal{F} = \mathcal{F}\,|p_\perp|$




- ✓ realistic model for dry frictional contact
- X non-conformity between continuous and discrete setting for $\mathcal{F}\,|p_\perp|$
- X analysis widely open

# Variational inequality (Tresca Friction)



- The differential model problem will be reformulated by means of the **Poincaré-Steklov operator** $\mathcal{S}$, defined by

$$\mathcal{S}(\mathbf{u}_{|\Gamma}) = \sigma[\mathbf{u}]_\Gamma \cdot \mathbf{n} = \mathbf{p} \qquad \text{on } \Gamma \times [0,T]$$

- Introducing functional set for displacement

$$\mathcal{C} = \{\mathbf{v}: (0,T] \times \Gamma \to \mathbb{R}^2 \,|\mathbf{v} = 0 \text{ a.e. on } [0,T] \times \Gamma_D, v_\perp \geq g \;\text{ a.c. on } [0,T] \times \Gamma_C\,\}$$

and defining the operators

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\sigma,\Gamma,(0,T]} := \int_0^T e^{-2\sigma t} \int_\Gamma \mathbf{u} \cdot \mathbf{v}\; d\Gamma_x\, dt, \qquad j(\mathbf{v}) := \int_0^T \int_{\Gamma_C} \boldsymbol{\mathcal{F}}\, \|\dot{\mathbf{v}}\|\, d\Gamma_x\, dt$$

we can prove the following result:

**Proposition**. The unilateral frictional contact problem can be equivalently expressed in the form of a **variational inequality,** namely:

*Find* $\mathbf{u} \in \mathcal{C}$ *such that*

$$\left\langle \mathcal{S}(\mathbf{u}), \boldsymbol{\partial}_{t,\|}(\mathbf{v}-\mathbf{u}) \right\rangle_{0,\Gamma_\Sigma,(0,T]} + j(\mathbf{v}) - j(\mathbf{u}) \geq \left\langle \mathbf{f}, \boldsymbol{\partial}_{t,\|}(\mathbf{v}-\mathbf{u}) \right\rangle_{0,\Gamma_\Sigma,(0,T]}, \qquad \forall \mathbf{v} \in \mathcal{C}$$

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# Mixed formulation



- Having considered the set of admissible Lagrange multipliers

$$M^+(\mathcal{F}) = \left\{ \boldsymbol{\mu} \in H^{\frac{1}{2}}\left([0,T]; \widetilde{H}^{-1/2}(\Gamma_C)\right)^2 \mid \langle \boldsymbol{\mu}, \boldsymbol{w}\rangle_{0,\Gamma_C,(0,T]} \le \left\langle \mathcal{F}, |w_{\|}|\right\rangle_{0,\Gamma_C,(0,T]} \ \forall\, \mathbf{w} \in H^{-\frac{1}{2}}\left([0,T]; \widetilde{H}^{1/2}(\Gamma_\Sigma)\right)^2, w_\perp \le 0 \right\}$$

in which the representative

$$\boldsymbol{\lambda} = \mathcal{S}(\mathbf{u}) - \mathbf{f} \in M^+(\mathcal{F})$$

is sought, the following result holds:

**Theorem**. The variational formulation of the contact problem with Tresca friction is equivalent to the following **mixed problem**:

*Find* $(\mathbf{u}, \lambda) \in H^{1/2}\left([0,T]; \widetilde{H}^{1/2}(\Gamma_\Sigma)\right)^2 \times M^+\,(\boldsymbol{\mathcal{F}})$ *such that*

$$\begin{cases} \langle \mathcal{S}(\mathbf{u}), \mathbf{v}\rangle_{0,\Gamma_\Sigma,(0,T]} - \langle \boldsymbol{\lambda}, \mathbf{v}\rangle_{0,\Gamma_C,(0,T]} = \langle \mathbf{f}, \mathbf{v}\rangle_{0,\Gamma_\Sigma,(0,T]}, & \forall\, \mathbf{v} \in H^{1/2}\left([0,T]; \widetilde{H}^{1/2}(\Gamma_\Sigma)\right)^2 \\ \left\langle \boldsymbol{\mu} - \boldsymbol{\lambda}, \boldsymbol{\partial}_{t,\|}\mathbf{u}\right\rangle_{0,\Gamma_C,(0,T]} \ge \langle g, \boldsymbol{\mu}_\perp - \boldsymbol{\lambda}_\perp\rangle_{0,\Gamma_C,(0,T]}, & \forall\, \boldsymbol{\mu} \in M^+(\boldsymbol{\mathcal{F}}) \end{cases}$$

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# Computation of Poincaré-Steklov operator $\mathcal{S}$



- Having introduced the **boundary integral operators** quartet:

$$[\mathcal{V}\mathbf{p}](\mathbf{x},t) = \int_{\Gamma} \mathbf{G}(\mathbf{x}-\boldsymbol{\xi};t) \star^{(t)} \mathbf{p}(\boldsymbol{\xi},t)\, d\Gamma_{\boldsymbol{\xi}}, \qquad [\mathcal{K}\mathbf{u}](\mathbf{x},t) = \int_{\Gamma} \mathbf{p}_{\xi}[\mathbf{G}]^{\top}(\mathbf{x}-\boldsymbol{\xi};t) \star^{(t)} \mathbf{u}(\boldsymbol{\xi},t)\, d\Gamma_{\boldsymbol{\xi}},$$

$$[\mathcal{K}^{*}\mathbf{p}](\mathbf{x},t) = \int_{\Gamma} \mathbf{p}_{x}[\mathbf{G}](\mathbf{x}-\boldsymbol{\xi};t) \star^{(t)} \mathbf{p}(\boldsymbol{\xi},t)\, d\Gamma_{\boldsymbol{\xi}}, \qquad [\mathcal{D}\mathbf{u}](\mathbf{x},t) = \int_{\Gamma} \mathbf{p}_{x}[\mathbf{p}_{\xi}[\mathbf{G}]^{\top}](\mathbf{x}-\boldsymbol{\xi};t) \star^{(t)} \mathbf{u}(\boldsymbol{\xi},t)\, d\Gamma_{\boldsymbol{\xi}},$$

  where **G** is the fundamental 2D elastodynamic solution, we can write the two classical BIEs:

$$\frac{1}{2}\mathbf{u} = \mathcal{V}\boldsymbol{p} - \mathcal{K}\mathbf{u} \quad \text{on} \quad \Gamma \times (0,T]$$

$$\frac{1}{2}\mathbf{p} = \mathcal{K}^{*}\mathbf{p} - \mathcal{D}\mathbf{u} \quad \text{on} \quad \Gamma \times (0,T]$$

- From these BIEs we obtain two equivalent expressions for $\mathcal{S}$:

SYMMETRIC FORMULATION

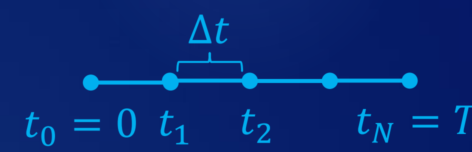
$$\mathcal{S} := \left(\mathcal{K}^{*} + \frac{1}{2}\mathcal{I}\right)\mathcal{V}^{-1}\left(\mathcal{K} + \frac{1}{2}\mathcal{I}\right) - \mathcal{D}$$

UNSYMMETRIC FORMULATION

$$\mathcal{S} := \mathcal{V}^{-1}\left(\mathcal{K} + \frac{1}{2}\mathcal{I}\right)$$

# Discretization by space-time BEM

- Discrete p.w. polynomial time spaces




$$V_{\Delta t}^{-1} = \left\{ v_{\Delta t} \in L^2([0,T]) : v_{\Delta t|_{[t_l,t_{l+1}]}} \in \mathcal{P}_0 \ \forall_{l=0}^{N} \right\}$$

$$v_{\Delta t,l}(t) = H[t - t_l] - H[t - t_{l+1}], \quad l = 0, \dots, N-1$$ p.w. constant basis

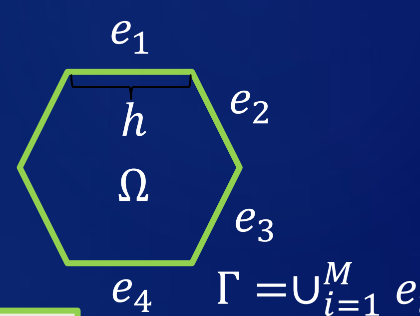
$$V_{\Delta t}^{0} = \left\{ r_{\Delta t} \in C^0([0,T]) : r_{\Delta t|_{[t_l,t_{l+1}]}} \in \mathcal{P}_1 \ \forall_{l=0}^{N}, r_{\Delta t}(0) = 0 \right\}$$

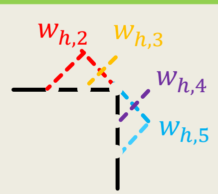
$$r_{\Delta t,l}(t) = H[t - t_l]\frac{t-t_l}{\Delta t} - H[t - t_{l+1}]\frac{t-t_{l+1}}{\Delta t}. \quad l = 0, \dots, N-1$$ p.w. linear basis

- Discrete p.w. polynomial spatial spaces

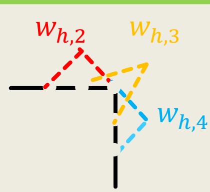
$$X_{h,\Gamma}^{-1} = \left\{ w_h \in L^2(\Gamma) : w_{h|_{e_i}} \in \mathcal{P}_s \ \forall_{i=1}^{M} \right\}$$

$$X_{h,\Gamma}^{0} = \left\{ w_h \in C^0(\Gamma) : w_{h|_{e_i}} \in \mathcal{P}_s \ \forall_{i=1}^{M} \right\}$$




p.w. linear basis for $L^2(\Gamma)$




p.w. linear basis for $C^0(\Gamma)$

- ➢ discretization of displacement and traction,
- ➢ $M_{H,\Delta T}^{+}(\mathcal{F}) = \left\{ \boldsymbol{\mu}_{H,\Delta T} \in \left( X_{H,\Gamma_C}^{-1} \otimes V_{\Delta T}^{-1} \right)^2 : \mu_{\perp,H,\Delta T} \geq 0 \text{ and } \left\| \mu_{\|,H,\Delta T} \right\| \leq \mathcal{F} \text{ on } \Gamma_C \times [0,T] \right\} \subset (\mathcal{F})$,
- ➢ discretization $\mathcal{S}_{h,\Delta t}$ of the operator $\mathcal{S}$.

# Uzawa Algorithm

- The **discretized mixed formulation** reads:

  *find* $\left(\mathbf{u}_{h,\Delta t}, \lambda_{H,\Delta T}\right) \in \left(X^0_{h,\Gamma_\Sigma} \otimes V^0_{\Delta t}\right)^2 \times M^+_{H,\Delta T}(\mathcal{F})$ *such that*

$$(*)\begin{cases} \left\langle \mathcal{S}_{h,\Delta t}(\mathbf{u}_{h,\Delta t}), \mathbf{v}_{h,\Delta t}\right\rangle_{0,\Gamma_\Sigma,(0,T]} - \left\langle \boldsymbol{\lambda}_{H,\Delta T}, \boldsymbol{v}_{h,\Delta t}\right\rangle_{0,\Gamma_C,(0,T]} = \left\langle \mathbf{f}, \mathbf{v}_{h,\Delta t}\right\rangle_{0,\Gamma_\Sigma,(0,T]} & \forall\, \mathbf{v}_{h,\Delta t} \in \left(X^0_{h,\Gamma_\Sigma} \otimes V^0_{\Delta t}\right)^2 \\ \left\langle \boldsymbol{\mu}_{H,\Delta T} - \boldsymbol{\lambda}_{H,\Delta T}, \boldsymbol{\partial}_{t,\parallel}\mathbf{u}_{h,\Delta t}\right\rangle_{0,\Gamma_C,(0,T]} \geq \left\langle g, \boldsymbol{\mu}_{\perp,h,\Delta t} - \boldsymbol{\lambda}_{\perp,H,\Delta T}\right\rangle_{0,\Gamma_C,(0,T]} & \forall\, \boldsymbol{\mu}_{H,\Delta T} \in M^+_{H,\Delta T}(\mathcal{F}) \end{cases}$$

- A **standard solver** for the discrete formulation defined above is given by the **Uzawa algorithm**:

> UZAWA ALGORITHM :
>
> 1) Choose $\rho > 0$. Set $k = 0$, $\boldsymbol{\lambda}^{(0)}_{H,\Delta T} = 0$.
> 2) WHILE stopping criterion not satisfied
>    - **I. solve:** $(*)$ for $\mathbf{u}^{(k)}_{h,\Delta t}$
>    - **II. project**:
>
> $$\boldsymbol{\lambda}^{(k+1)}_{H,\Delta T} = Pr_C\left(\boldsymbol{\lambda}^{(k)}_{H,\Delta T} - \rho\left(\boldsymbol{\partial}_{t,\parallel}\mathrm{u}^{(k)}_{h,\Delta t} - \mathbf{g}\right)\right)$$
>
>    - **III. update**: $k = k + 1$
>
>    END WHILE

non-penetration of the interface:

$$Pr_C(\mathbf{w})_\perp = \max\{w_\perp, 0\}$$

frictional constraint:

$$Pr_C(\mathbf{w})_\perp = \begin{cases} w_\parallel, & if\ \ |w_\parallel| \leq \mathcal{F} \\ \mathcal{F}\dfrac{w_\parallel}{|w_\parallel|}, & if\, |w_\parallel| > \mathcal{F} \end{cases}$$

# Theoretical results



- Under assumption of a positive definite discrete operator $\mathcal{S}_{h,\Delta t}$ and for null friction forces ($\mathcal{F} = 0$), convergence of Uzawa algorithm has been proved. (1)
- In case of Tresca friction we prove **a priori error estimates**:

**Theorem.** (2)

- $\mathcal{S}_{h,\Delta t}$ is positive definite
- $\frac{\max\{h,\Delta t\}}{\min\{H,\Delta T\}} < c$ for a small c > 0

⇒ upper bounds for the discretization errors $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{H,\Delta T}\|_{H^*}$ and $\|\mathbf{u} - \mathbf{u}_{h,\Delta t}\|_{H^\circ}$ in suitable Sobolev norms.

**Remarks**:

- Existence of solutions only known for simple toy problems (wave equation in half-space, no friction).
- if the solution $(\mathbf{u}, \lambda)$ exists and is sufficiently regular, we obtain convergence rates which are optimal in space, but suboptimal in time.
- **Rigorous analysis of the more realistic Coulomb friction remains widely open**, even for stationary contact problems, because constraints depend on solution.

1: A. Aimi, G. Di Credico, H. Gimperlein, *CMAME, 415 (2023)*, 116296.
2: A. Aimi, G. Di Credico, H. Gimperlein, *CMAME, 427 (2024)*, 117066.

# Resolution step (∗) of the Uzawa algorithm

*Find* $\mathbf{u}_{h,\Delta t} \in \left(X^0_{h,\Gamma_\Sigma} \otimes V^0_{\Delta t}\right)^2$ *such that*

$$\left\langle \mathcal{S}_{h,\Delta t}(\mathbf{u}_{h,\Delta t}), \dot{\mathbf{v}}_{h,\Delta t} \right\rangle_{L^2(\Gamma_\Sigma \times (0,T])} = \left\langle \tilde{\mathbf{f}}, \dot{\mathbf{v}}_{h,\Delta t} \right\rangle_{L^2(\Gamma \times (0,T])} \quad \forall\, \mathbf{v}_{h,\Delta t} \in \left(X^0_{h,\Gamma_\Sigma} \otimes V^0_{\Delta t}\right)^2$$

⬇

$$\mathbb{S}\mathbf{X} = \tilde{\mathbf{F}} \quad \text{where} \quad \mathbb{S} = \begin{pmatrix} \mathbb{S}^{(0)} & 0 & 0 & 0 & 0 \\ \mathbb{S}^{(1)} & \mathbb{S}^{(0)} & 0 & 0 & 0 \\ \mathbb{S}^{(2)} & \mathbb{S}^{(1)} & \mathbb{S}^{(0)} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \mathbb{S}^{(N-1)} & \mathbb{S}^{(N-2)} & \mathbb{S}^{(N-3)} & \dots & \mathbb{S}^{(0)} \end{pmatrix}$$

## SYMMETRIC FORMULATION

$$\begin{cases} \left\langle \mathcal{V}\mathbf{p}_{h,\Delta t} - \left(\mathcal{K} + \frac{1}{2}\right)\mathbf{u}_{h,\Delta t}, \dot{\boldsymbol{\psi}}_{h,\Delta t} \right\rangle = 0 \\ \left\langle \left(\mathcal{K}^* + \frac{1}{2}\right)\mathbf{p}_{h,\Delta t} - \mathcal{D}\mathbf{u}_{h,\Delta t}, \dot{\mathbf{v}}_{h,\Delta t} \right\rangle = \left\langle \tilde{\mathbf{f}}, \dot{\mathbf{v}}_{h,\Delta t} \right\rangle \end{cases}$$

$$\mathbb{S}^{(0)} = \begin{pmatrix} \mathbb{E}^{(0)}_{\mathcal{V}} & -\mathbb{E}^{(0)}_{\mathcal{K}} - \frac{1}{2}\mathbb{M} \\ {\mathbb{E}^{(0)}_{\mathcal{K}}}^\top + \frac{1}{2}\mathbb{M} & -\mathbb{E}^{(0)}_{\mathcal{D}} \end{pmatrix}$$

$$\mathbb{S}^{(l)} = \begin{pmatrix} \mathbb{E}^{(l)}_{\mathcal{V}} & -\mathbb{E}^{(l)}_{\mathcal{K}} \\ {\mathbb{E}^{(l)}_{\mathcal{K}}}^\top & -\mathbb{E}^{(l)}_{\mathcal{D}} \end{pmatrix} \quad \forall_{l=1}^{N-1}$$

## UNSYMMETRIC FORMULATION

$$\begin{cases} \left\langle \mathcal{V}\mathbf{p}_{h,\Delta t} - \left(\mathcal{K} + \frac{1}{2}\right)\mathbf{u}_{h,\Delta t}, \dot{\boldsymbol{\psi}}_{h,\Delta t} \right\rangle = 0 \\ \left\langle \mathbf{p}_{h,\Delta t}, \dot{\mathbf{v}}_{h,\Delta t} \right\rangle = \left\langle \tilde{\mathbf{f}}, \dot{\mathbf{v}}_{h,\Delta t} \right\rangle \end{cases}$$

$$\mathbb{S}^{(0)} = \begin{pmatrix} \mathbb{E}^{(0)}_{\mathcal{V}} & -\mathbb{E}^{(0)}_{\mathcal{K}} - \frac{1}{2}\mathbb{M} \\ \mathbb{M}^\top & 0 \end{pmatrix}$$

$$\mathbb{S}^{(l)} = \begin{pmatrix} \mathbb{E}^{(l)}_{\mathcal{V}} & -\mathbb{E}^{(l)}_{\mathcal{K}} \\ 0 & 0 \end{pmatrix} \quad \forall_{l=1}^{N-1}$$

$$u_{i,h,\Delta t}(\mathbf{x},t) = \sum_{l=0}^{N-1} \sum_{m=1}^{M_{\mathbf{u}}} u_{i,l,m}\, w_{h,m}(\mathbf{x}) r_{\Delta t,l}(t), \qquad p_{i,h,\Delta t}(\mathbf{x},t) = \sum_{l=0}^{N-1} \sum_{m=1}^{M_{\mathbf{p}}} p_{i,l,m}\, w_{h,m}(\mathbf{x}) v_{\Delta t,l}(t)$$

# Algebraic form of the Uzawa algorithm



$$\lambda_{i,h,\Delta t}(\mathbf{x},t) = \sum_{l=0}^{N-1} \sum_{m=1}^{M_\lambda} \lambda_{i,l,m}\, w_{h,m}(\mathbf{x}) v_{\Delta t,l}(t) \qquad \text{discretization of } \lambda$$

- $\boldsymbol{\Lambda} = (\boldsymbol{\Lambda}_{(0)}, \boldsymbol{\Lambda}_{(1)}, \dots, \boldsymbol{\Lambda}_{(N-1)})$ with $\boldsymbol{\Lambda}_{(l)} = \left(\lambda_{1,l,1}, \dots, \lambda_{1,l,M_{\boldsymbol{\lambda}}}, \lambda_{2,l,1}, \dots, \lambda_{2,l,M_{\boldsymbol{\lambda}}}\right)^\top$
- $\mathbf{X} = (\mathbf{X}_{(0)}, \mathbf{X}_{(1)}, \dots, \mathbf{X}_{(N-1)})$ where $\mathbf{X}_{(l)} = \left(\mathbf{U}_{(l)}, \mathbf{P}_{(l)}\right)^\top$ with

$\mathbf{U}_{(l)} = \left(u_{1,l,1}, \dots, u_{1,l,M_{\mathbf{u}}}, u_{2,l,1}, \dots, u_{2,l,M_{\mathbf{u}}}\right)^\top$, $\mathbf{P}_{(l)} = \left(p_{1,l,1}, \dots, p_{1,l,M_{\mathbf{p}}}, p_{2,l,1}, \dots, p_{2,l,M_{\mathbf{p}}}\right)^\top$

UZAWA ITERATIONS :

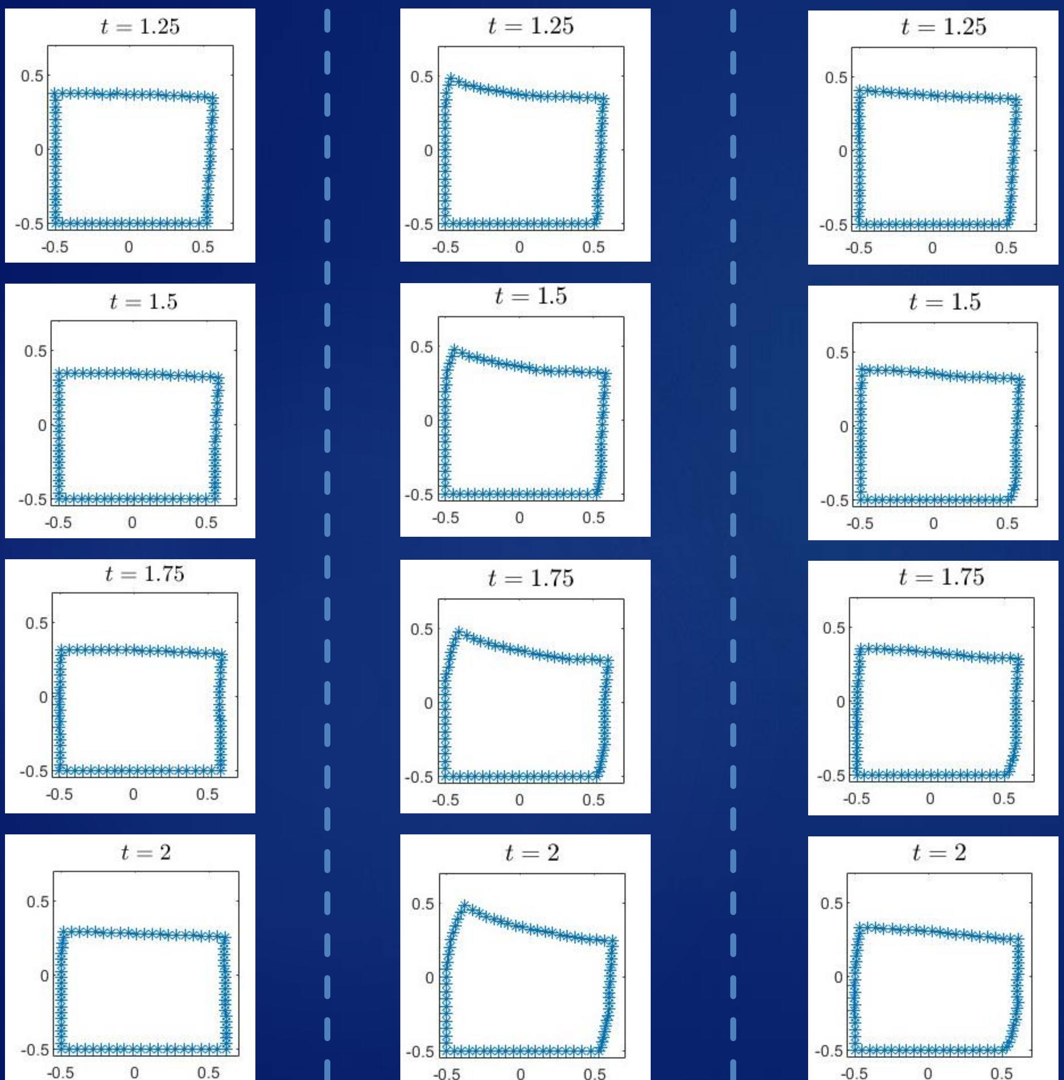
1) Fix $\rho > 0$ and $\epsilon > 0$. Construct $\mathbb{S}$, $\mathbf{F}$. Set $k = 0, \boldsymbol{\Lambda}^{(0)} = \mathbf{0}$ and $\boldsymbol{\Lambda}^{(-1)} = \mathbf{1}$.
2) WHILE $\dfrac{\left\|\boldsymbol{\Lambda}^{(k)} - \boldsymbol{\Lambda}^{(k-1)}\right\|_2}{\left\|\boldsymbol{\Lambda}^{(k)}\right\|_2} > \epsilon$

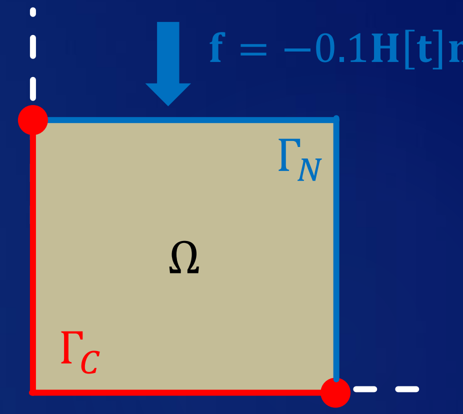
   **I. solve**: $\mathbb{S}\mathbf{X}^{(k)} = \mathbf{F} + \mathbb{M}^* \boldsymbol{\Lambda}^{(k)}$
   
   **II. extract**: $\mathbf{U}^{(k)}$ from $\mathbf{X}^{(k)}$
   
   **III. compute**: $\boldsymbol{\Lambda}^{(k+1)} = pr_C\left(\boldsymbol{\Lambda}^{(k)} - \rho \widetilde{\mathbb{M}}(\mathbf{U}^{(k)} - \mathbf{G})\right)$
   
   **IV. update**: $k = k + 1$
   
   END WHILE

$$\left(pr_C(\mathbf{w})\right)_j = \begin{cases} \max\{w_j, 0\}, & j \in J_\perp \\ w_j & if\ |w_j| \le \mathcal{F}_j \text{ and } j \in J_\parallel \\ \mathcal{F}_j \dfrac{w_j}{|w_j|} & if\ |w_j| > \mathcal{F}_j \text{ and } j \in J_\parallel \end{cases}$$

# Numerical tests: compressing a square

no friction $\mathcal{F} = 0$ | Tresca friction $\mathcal{F} = 0.05$ | Coulomb friction $\mathcal{F} = 0.5$

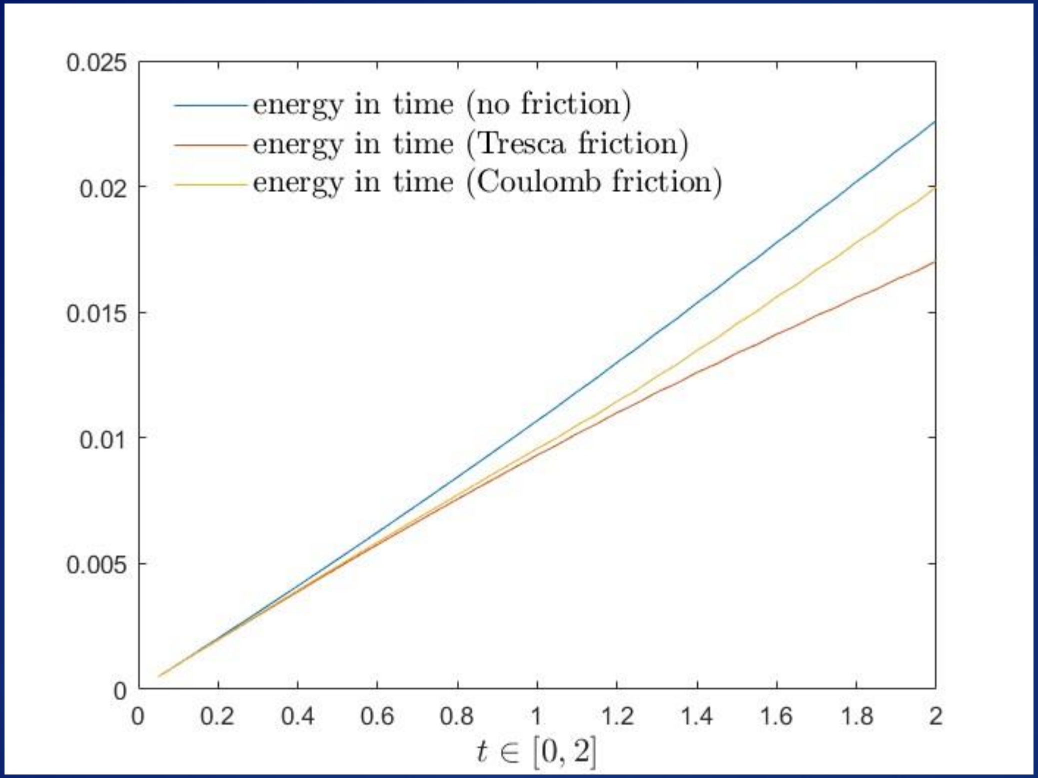


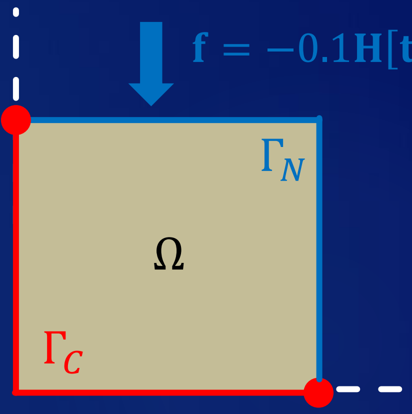




- constant traction applied to the top of the box $\Omega = [-0.5,0.5] \times [-0.5,0.5]$
- physical and mesh parameters: $c_P = 1, c_S = 0.5$, $h = \Delta t = 0.05$

# Numerical tests: compressing a square





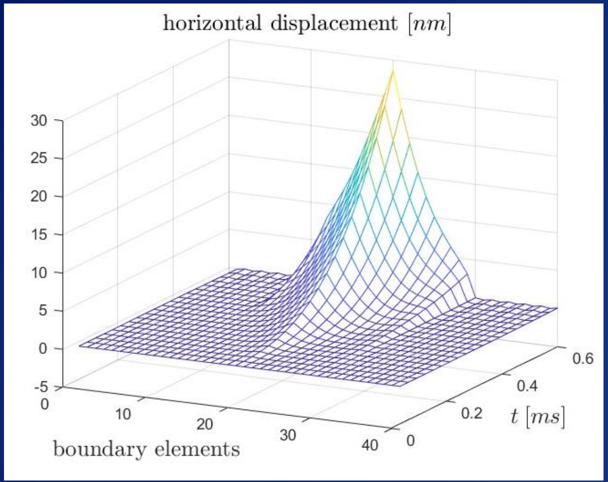



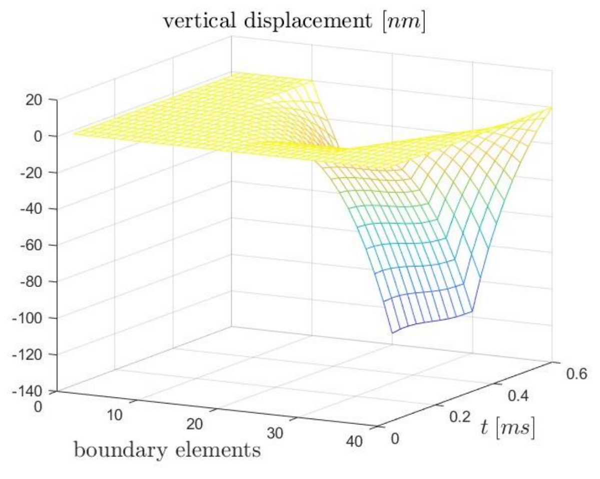

- constant traction applied to the top of the box $\Omega = [-0.5,0.5] \times [-0.5,0.5]$
- physical and mesh parameters: $c_P = 1, c_S = 0.5$, $h = \Delta t = 0.05$

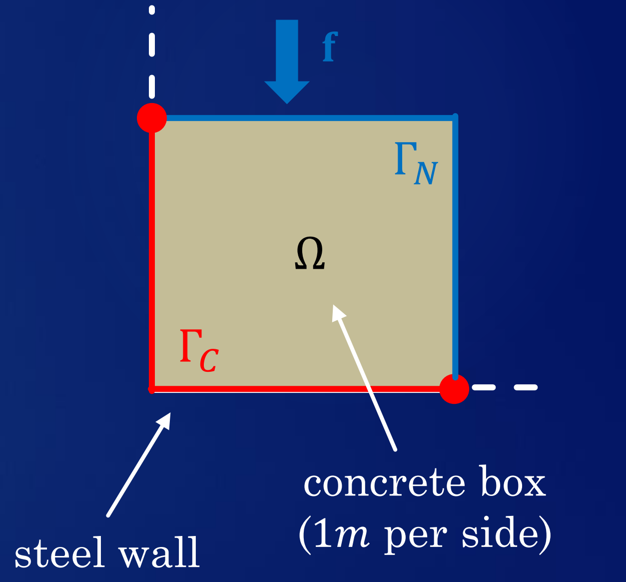
linear increase of the energy for short times, before Tresca and Coulomb frictional contact dissipates some of the introduced energy, compared to the case without friction.

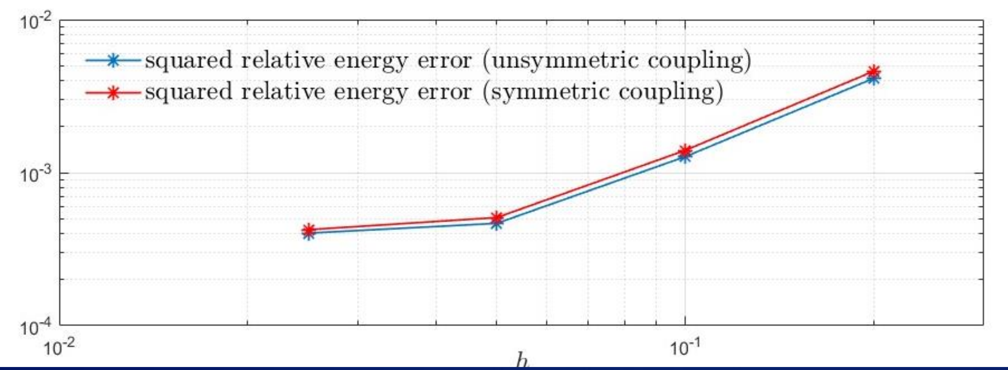
# Numerical tests: compressing a square - concrete against steel

- constant traction applied to the top of the concrete box: $\mathbf{f} = -4\tanh\left(\left(\frac{\mathbf{t}}{\mathbf{15}}\right)^{\mathbf{2}}\right)\mathbf{n}\,\frac{kg}{m\cdot ms^2}$
- physical parameters of the box: $c_P = 3.253\,\frac{m}{ms}, c_S = 1.992 m/ms$
- Coulomb friction $\mathcal{F}_c = 0.3$ corresponds to the interaction between concrete and steel



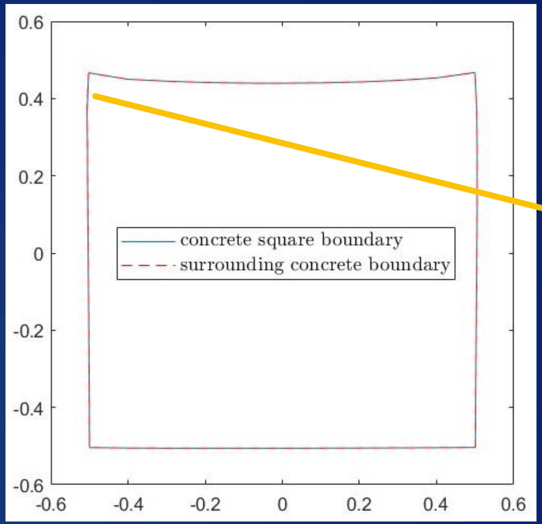



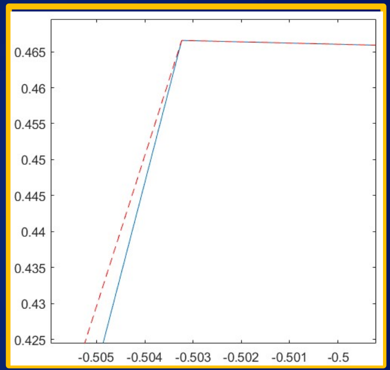

- The squared energy error for both couplings decays with similar slope

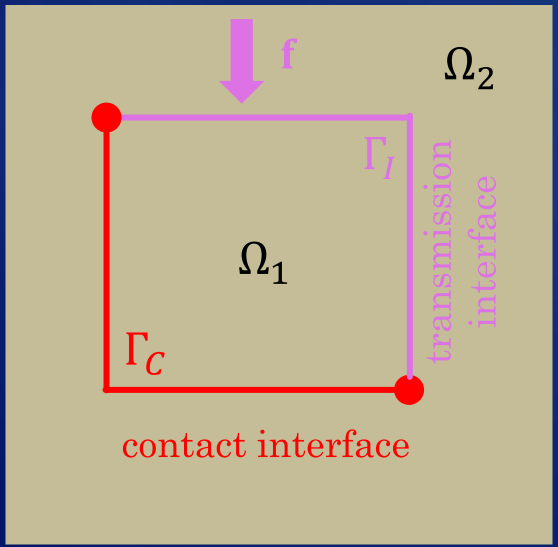
# Numerical tests: two-body contact – concrete against concrete

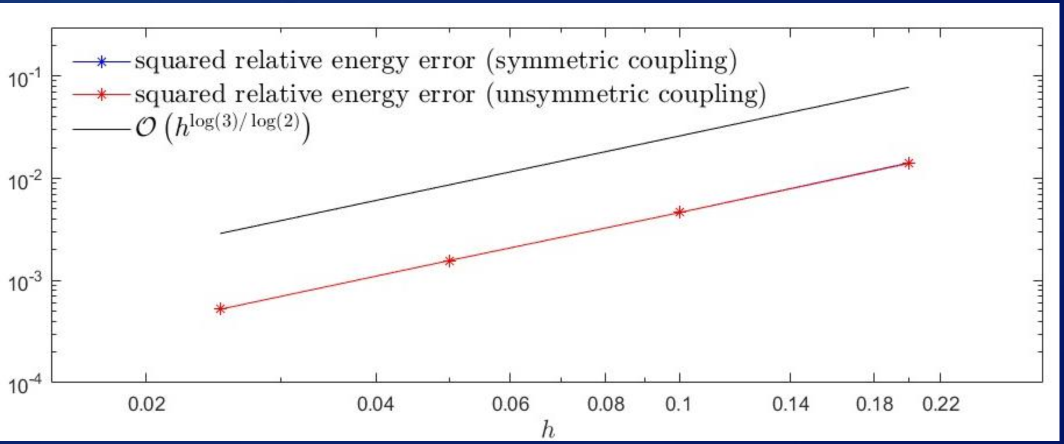


- concrete box $\Omega_1$ ($1m$ per side) embedded in concrete material $\Omega_2$
- constant traction applied to the top of $\Omega_1$: $\mathbf{f} = -4\tanh\left(\left(\frac{\mathbf{t}}{15}\right)^2\right)\mathbf{n}\,\frac{kg}{m\cdot ms^2}$
- Coulomb friction $\mathcal{F}_c = 0.75$ corresponds to the concrete – concrete interaction



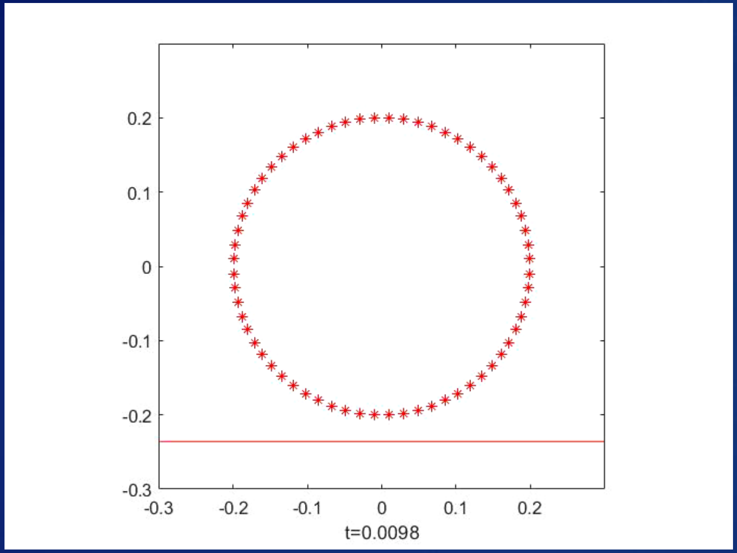


# Numerical tests: tennis ball

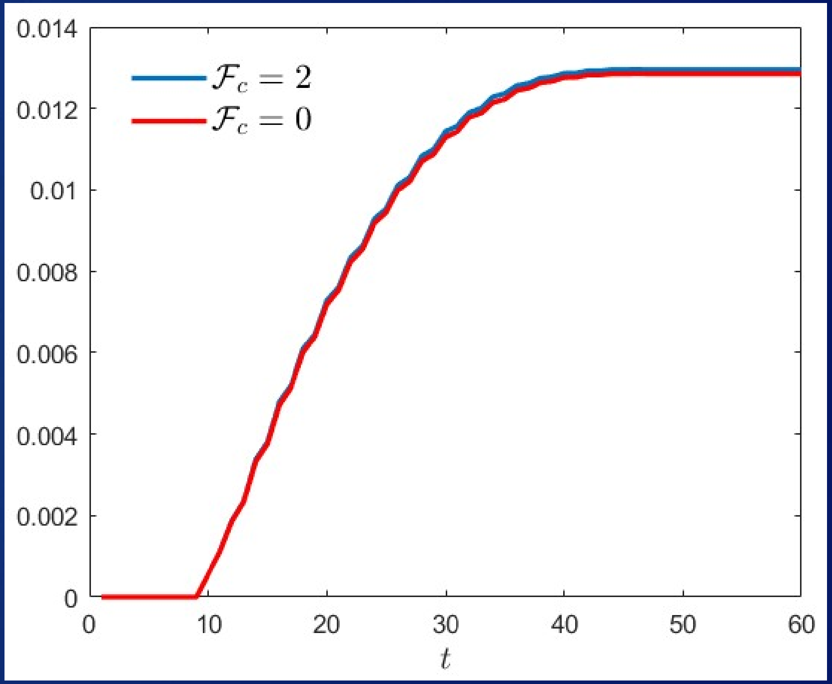


- The horizontal contact surface starts to push up gently the disk from the bottom
- No external traction and forces: after the released the disk flies up with constant velocity
- physical and mesh parameters: $c_P = 2, c_S = 1, h = 2\Delta t (\simeq 0.04)$




Dynamical deformation of the disk:

- test run with symmetric form of the Poincaré-Steklov operator
- Coulomb type friction $\mathcal{F}_c = 2$




Energy in time with and without friction

# Conclusions



- ✓ We investigated space-time boundary elements to solve frictional contact problems in linear elastodynamics
- ✓ Boundary elements provide a natural and efficient discretization approach
- ✓ We have an a priori estimate for the error of the numerical solution
- ✓ Numerical experiments in two space dimensions confirm the method beyond this ideal (two-sided contact and realistic friction laws)


- A. Aimi, G. Di Credico, H. Gimperlein: **Space–time boundary elements for frictional contact in elastodynamics**, *CMAME, 427 (2024)*, 117066.
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