

## Boundary Elements and Mesh Refinements for the Wave Equation

\*David Stark<sup>1</sup>, Heiko Gimperlein<sup>1</sup>

<sup>1</sup>Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS

\*ds221@hw.ac.uk

### ABSTRACT

We discuss time domain boundary element methods for singular geometries, in particular graded meshes and adaptive mesh refinements. First, we discuss edge and corner singularities for a Dirichlet problem for the wave equation. Time independent graded meshes lead to efficient approximations, as confirmed by numerical experiments for wave scattering from screens. We briefly discuss adaptive mesh refinement procedures based on a posteriori error estimates. A modified MOT scheme provides an efficient preconditioner (or stand-alone solver) for the space-time systems obtained for the Galerkin discretisations.

**Key Words:** Wave equation; time-domain boundary element method; adaptive mesh refinements; graded meshes.

### 1. Introduction

Boundary element methods provide an efficient, extensively studied numerical scheme for time-independent or time-harmonic scattering and emission problems. Unlike finite element discretisations, they reduce the computation from the three dimensional domain to its two dimensional boundary. Recently, boundary elements have been explored for the simulation of transient phenomena, with applications e.g. to environmental noise [2] or electromagnetic scattering [7]. Galerkin time domain boundary element methods prove to be stable and accurate in long-time computations and are competitive with frequency domain methods for realistic problems [3].

In this talk we discuss recent work on adaptive mesh refinements and graded meshes for singular geometries, as motivated by the sound emission on tires [2].

### 2. Problem description and time domain BEM

We consider the wave equation outside a scatterer  $\Omega^-$  in  $\mathbb{R}^3$ , where  $\Omega^-$  is a bounded polygon or a screen with connected complement  $\Omega = \mathbb{R}^3 \setminus \Omega^-$ . The acoustic sound pressure field  $u$  due to an incident field or sources on  $\Gamma = \partial\Omega$  satisfies the linear wave equation for  $(t, \mathbf{x}) \in \mathbb{R} \times \Omega$ :

$$\partial_t^2 u(t, \mathbf{x}) - \Delta u(t, \mathbf{x}) = 0$$

with Dirichlet boundary conditions  $u(t, \mathbf{x}) = f(t, \mathbf{x})$  for  $\mathbf{x} \in \Gamma$ , and  $u(t, \mathbf{x}) = 0$  for  $t \leq 0$ .

A single-layer ansatz for  $u$ ,

$$u(t, \mathbf{x}) = \int_{\Gamma} \frac{\phi(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|} ds_{\mathbf{y}}, \quad (1)$$

results in an equivalent weak formulation as an integral equation of the first kind in space-time anisotropic Sobolev spaces [6, 3]:

Find  $\phi \in H_{\sigma}^1(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma))$  such that for all  $\psi \in H_{\sigma}^1(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma))$

$$\int_0^{\infty} \int_{\Gamma} (V\phi(t, \mathbf{x})) \partial_t \psi(t, \mathbf{x}) ds_x d_{\sigma}t = \int_0^{\infty} \int_{\Gamma} f(t, \mathbf{x}) \partial_t \psi(t, \mathbf{x}) ds_x d_{\sigma}t, \quad (2)$$

where  $d_{\sigma}t = e^{-2\sigma t} dt$  and

$$V\phi(t, \mathbf{x}) = \int_{\Gamma} \frac{\phi(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|} ds_{\mathbf{y}}.$$

A theoretical analysis requires  $\sigma > 0$ , but practical computations use  $\sigma = 0$ .

We study time dependent boundary element methods to solve (2), based on approximations by piecewise polynomial ansatz and test functions from the space  $V_{h,\Delta t}^{p,q}$  spanned by

$$\phi_i(t, \mathbf{x}) = \tilde{\Lambda}_i(t)\Lambda_i(\mathbf{x}) . \quad (3)$$

Here,  $\Lambda_i$  a piecewise polynomial shape function of degree  $p$  in space and  $\tilde{\Lambda}_i$  a corresponding shape function of degree  $q$  in time. For  $p \geq 1$ , resp.  $q \geq 1$ , the shape functions are assumed to be continuous.

We obtain a numerical scheme for the weak formulation (2): Find  $\phi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$  such that for all  $\psi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$

$$\int_0^\infty \int_\Gamma (V\phi_{h,\Delta t}(t, \mathbf{x})) \partial_t \psi_{h,\Delta t}(t, \mathbf{x}) ds_x dt = \int_0^\infty \int_\Gamma f(t, \mathbf{x}) \partial_t \psi_{h,\Delta t}(t, \mathbf{x}) ds_x dt . \quad (4)$$

From  $\phi_{h,\Delta t}$ , the sound pressure  $u_{h,\Delta t}$  is obtained in  $\Omega$  by evaluating the integral in (1) numerically.

### 3. Error estimate and adaptive mesh refinements

Computable error indicators are a key ingredient to design adaptive mesh refinements. We recall from [3]:

**Theorem:** Let  $\phi, \phi_{h,\Delta t} \in H_0^1([0,T], H^{-\frac{1}{2}}(\Gamma))$  be the solutions to (2) resp. (4). Assume that  $R = \dot{f} - V\dot{\phi}_{h,\Delta t} \in H^0([0,T], H^1(\Gamma))$ . Then

$$\|\phi - \phi_{h,\Delta t}\|_{0, -\frac{1}{2}, \Gamma}^2 \lesssim \max\{\Delta t, h\} (\|\partial_t R\|_{L^2([0,T], L^2(\Gamma))} + \|\nabla R\|_{L^2([0,T], L^2(\Gamma))})^2.$$

The error indicators  $\eta(\Delta)$  lead to

#### Adaptive Algorithm:

Input: Mesh  $\mathcal{T} = \mathcal{T}_0$ , refinement parameter  $\theta \in (0, 1)$ , tolerance  $\epsilon > 0$ , data  $f$ .

1. Solve  $V\dot{\phi}_{h,\Delta t} = \dot{f}$  on  $\mathcal{T}$ .
2. Compute the error indicators  $\eta(\Delta)$  in each triangle  $\Delta \in \mathcal{T}$ .
3. Find  $\eta_{max} = \max_{\Delta} \eta(\Delta)$ .
4. Stop if  $\sum_i \eta^2(\Delta_i) < \epsilon^2$ .
5. Mark all  $\Delta \in \mathcal{T}$  with  $\eta(\Delta_i) > \theta \eta_{max}$ .
6. Refine each marked triangle into 4 new triangles to obtain a new mesh  $\mathcal{T}$
7. Go to 1.

Output: Approximation of  $\dot{\phi}$ .

### 4. Graded meshes

The realistic scattering and diffraction of waves in  $\mathbb{R}^3$  is crucially affected by geometric singularities of the scatterer, as solutions of the wave equation exhibit well-known singularities at non-smooth boundary points of the domain. For such geometric singularities, graded meshes adapted to the geometry are known to provide optimal convergence rates for time independent problems. We have generalised to time-independent theory to the transient setting [4] and illustrate the improved convergence rates in numerical experiments below.

Our computations are mainly conducted on the square  $[-1, 1]^2$ . To define  $\beta$ -graded meshes, due to symmetry, it suffices to consider a  $\beta$ -graded mesh on  $[-1, 0]$ . We define  $y_k = x_k = -1 + (\frac{k}{N_l})^\beta$  for  $i = 1, \dots, N_l$  and for a constant  $\beta \geq 0$ . The nodes of the  $\beta$ -graded mesh are therefore  $(x_k, y_l)$ ,  $k, l = 1, \dots, N_l$ . We note that for  $\beta = 1$  we would have a uniform mesh.

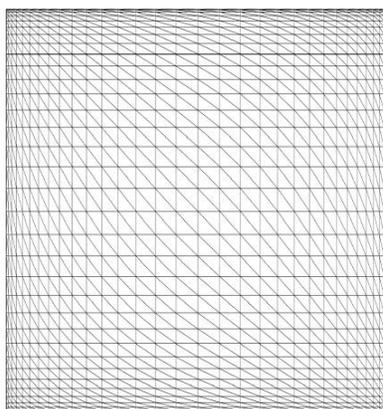


Figure 1:  $\beta$ -graded mesh with  $n = 20$  triangles per side and  $\beta = 2$

### 5. Numerical results

1) If  $\Gamma$  is a screen, the density  $\phi$  exhibits edge and corner singularities. Motivated by recent work by Müller and Schwab for 2d FEM, in [4] we adapt a classical analysis by von Petersdorff for time-independent problems and obtain the precise singular behaviour of  $\phi$  near  $\partial\Gamma$ :  $\phi(t, \mathbf{x}) \sim \text{dist}(\mathbf{x}, \partial\Gamma)^{-1/2}$  near an edge,  $\phi(t, \mathbf{x}) \sim \text{dist}(\mathbf{x}, \partial\Gamma)^{-0.703}$  near a right-angled corner.

Time-independent graded meshes provide a quasi-optimal approximation of these singularities. The numerical experiment depicted in Figure 2 compares the convergence in energy norm on graded and uniform meshes for  $\Gamma = [0, 1]^2 \times \{0\}$  and illustrates the theoretically predicted convergence of order  $DOF^{-1}$ , resp.  $\sim DOF^{-1/2}$ . We present an application to the singular horn geometry between a vibrat-

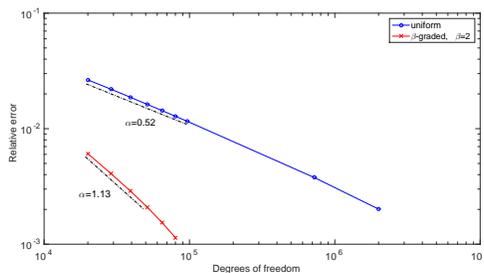


Figure 2: Energy convergence for graded vs. uniform meshes.

ing tire and the road.

2) We use provably reliable residual error indicators, as well as heuristic ZZ and hierarchical indicators to steer adaptive mesh refinements. Figure 3 shows that the residual indicators converge at the same rate as the energy error for an example problem with  $\Gamma = S^2$ . We briefly recall the theoretical results on reliability and (weak) efficiency of the residual error indicators (see [3]) and compare the adaptive methods obtained from the different error indicators.

3) To obtain provably stable methods and a rigorous error analysis, we require conforming Galerkin discretisations. In general, the discretised equation (4) corresponds to a lower Hessenberg linear system in space-time, with one band above the diagonal. Motivated by adaptivity and  $C^\infty$  temporal basis functions, there has been much recent interest in works by Sauter-Veit, Merta et al. , Schanz and others in efficient solvers. We present an approximate time-stepping scheme [5], based on extrapolation, which becomes exact for  $\Delta t \rightarrow 0$ . It may be used as either a preconditioner or standalone solver. Table 1

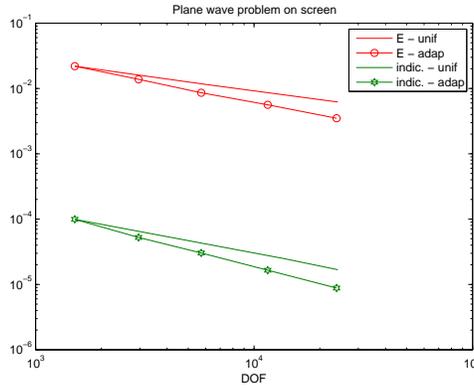


Figure 3: Uniform vs adaptive for plane wave problem on  $[0, 0.5]^2$  screen.

Geometry	DOF	Energy	MOT	Error
sphere	320	8.5692	8.5470	.26%
	1280	8.6059	8.6059	$\ll 1\%$
icosahed.	320	20.538	21.480	4.6%
	1280	19.879	20.143	1.3%
screen	288	0.4233	0.4497	6.2%
	1250	0.4589	0.4716	2.8%

Table 1: relative errors in energy: modified MOT vs. GMRES with residual  $10^{-9}$ .

compares one step of this method to the (essentially exact) solution of the space-time system as obtained from GMRES.

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### References

- [1] A. Bamberger, T. Ha Duong, *Formulation variationnelle espace-temps pour le calcul par potentiel retardé de la diffraction d'une onde acoustique*, Math. Meth. Appl. Sci. 8 (1986), 405-435, 598-608.
- [2] L. Banz, H. Gimperlein, Z. Nezhi, E. P. Stephan, *Time domain BEM for sound radiation of tires*, Comput. Mech. **58** (2016), 45-57.
- [3] H. Gimperlein, M. Maischak, E. P. Stephan, *Adaptive time domain boundary element methods and engineering applications*, survey, Journal of Integral Equations and Applications, to appear (2016).
- [4] H. Gimperlein, F. Meyer, C. Özdemir, D. Stark, E. P. Stephan, *Boundary elements with mesh refinements for the wave equation*, in preparation (2017).
- [5] H. Gimperlein, D. Stark, *Extrapolation as preconditioner for time domain boundary element methods*, preprint (2016).
- [6] F.-J. Sayas, *Retarded Potentials and Time Domain Boundary Integral Equations: a road-map*, Springer Series in Comp. Math. 50, Springer, 2016.
- [7] A. E. Yilmaz, J.-M. Jin, E. Michielssen, *Time domain adaptive integral method for surface integral equations*, IEEE Trans. Antennas Propagation 52 (2004), 2692-2708.