Adaptive boundary elements for wave equations

Heiko Gimperlein[1]

(with A. Aimi[2], T. Chaumont-Frelet[3], G. Di Credico[2], C. Guardasoni[2], I. Labarca-Figueroa[4], J. Nick[5], E. P. Stephan[6])

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Söllerhaus, September 27, 2025

Chaumont-Frelet, HG, Labarca-Figueroa, Nick, arXiv:2509.04217

Aimi, Di Credico, HG, Guardasoni, *A space-time adaptive boundary element method for the wave equation*, preprint (2025).

HG, Özdemir, Stark, Stephan, Numer. Math. 146 (2020), 239-280.

Wave equation

$u = u(t,x)$ sound pressure

$$\begin{aligned} \partial_t^2 u - \Delta u &= 0 \quad && \text{in } \mathbb{R}_t \times \mathbb{R}^n_x \setminus \Omega \\ u &= 0 && \text{for } t \le 0\ . \end{aligned}$$

Simple: Dirichlet boundary conditions $u = g$ on $\Gamma = \partial\Omega$.

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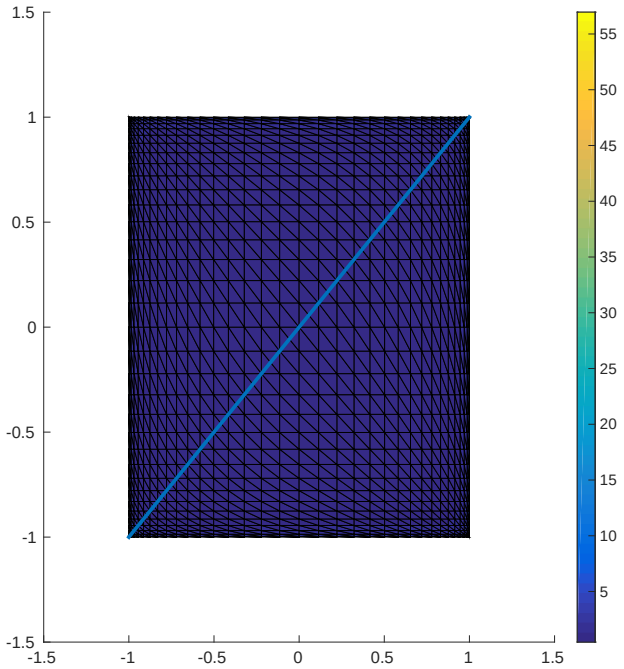
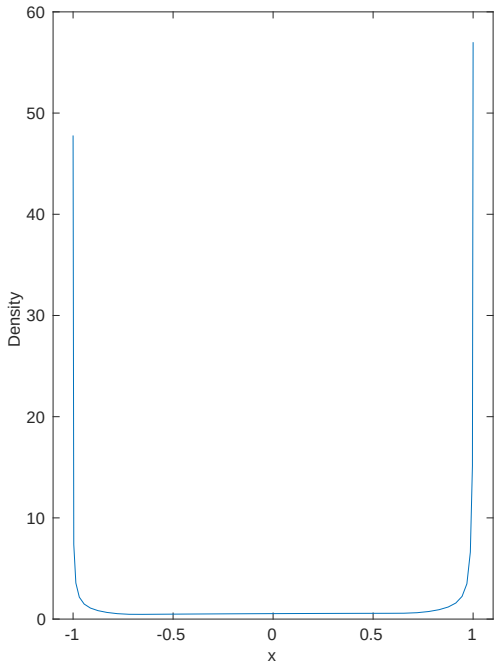
Simple: Dirichlet boundary conditions $u = g$ on $\Gamma = \partial\Omega$.

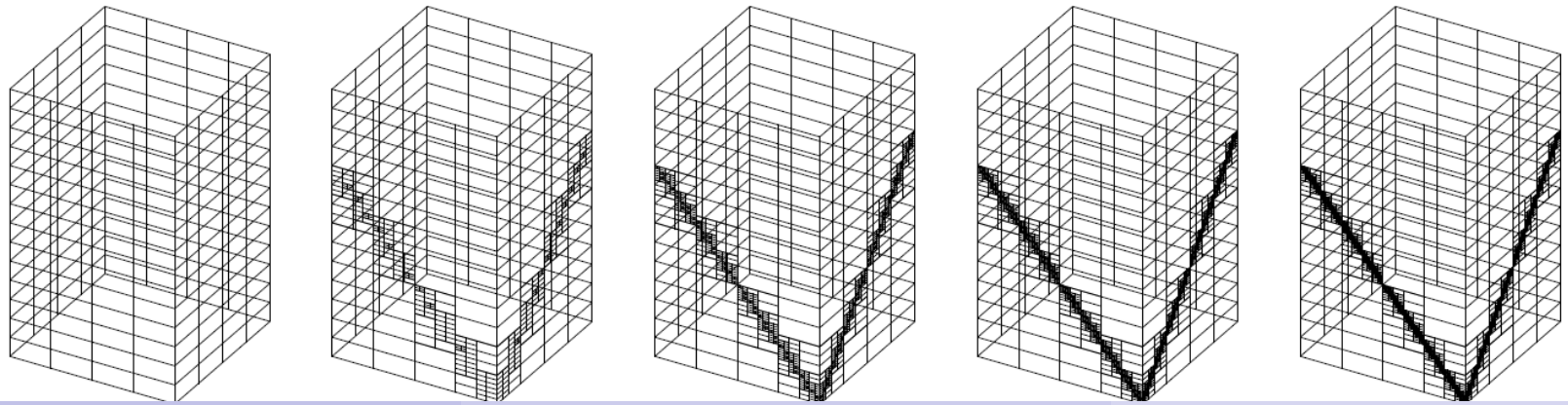
Space-time boundary integral formulation of Dirichlet problem ($n = 3$):

$$\mathcal{V}\phi(t,x) = \int_\Gamma \frac{\phi(x', t - |x - x'|)}{4\pi|x - x'|}\, d\Gamma_{x'} = (\mathcal{K} + \tfrac{1}{2})g(t,x)$$

What's this talk about?







Related work: old and new

$$u = u(t,x) \quad \text{sound pressure}$$
$$\partial_t^2 u - \Delta u = 0 \qquad \text{in } \mathbb{R}_t \times \mathbb{R}_x^n \setminus \Omega$$
$$u = 0 \qquad \text{for } t \le 0 \ .$$

Singular solutions:

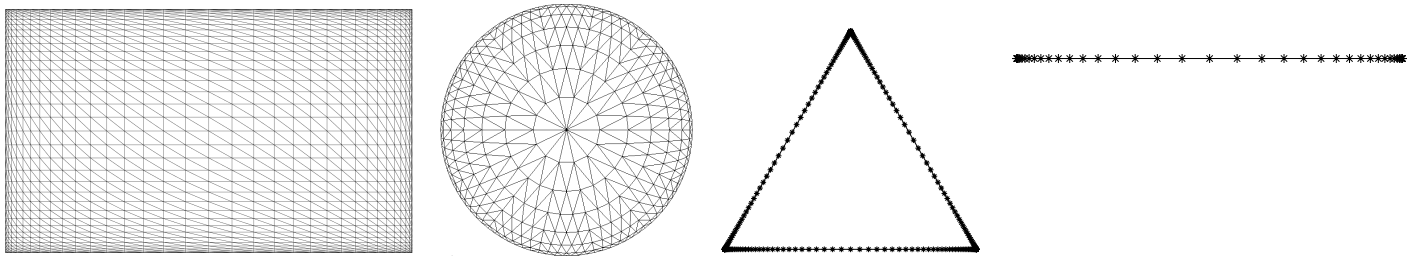
- Edge/corner/geometric singularities.
- Sharp travelling wave crests.
- (Nonlinear) contact problems.

local mesh refinements + adaptive BEM:
variable Δt: Sauter, Schanz, Banjai, . . . (gCQ), Veit, Steinbach, Zank, Löscher, Hoonhout, Urzua-Torres, . . . (1d)
variable Δx: Abboud
space-time (2d, brute force): Gläfke, Maischak

FEM on graded meshes: Müller, Schwab, Moiola, Perugia . . .
adaptive FEM: Chaumont-Frelet (SISC '23), Ern, . . . (variable Δx)

Outline

- Space-time boundary elements: basic set up
- Singularities and approximation near edges / corners
- Residual a posteriori error estimates for space-time BEM and CQ
- Space- and space-time adaptive mesh refinements



(a) mesh 1 (b) mesh 4 (c) mesh 7 (d) mesh 10 (e) mesh 13

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Space-time boundary integral formulation of Dirichlet problem (3d):

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Set-up a la Bamberger–Ha Duong

$$\mathcal{V}\phi(x,t) = \left(\mathcal{K} + \frac{1}{2}\right) g(x,t) =: f$$

space–time anisotropic Sobolev spaces $H^r_\sigma(\mathbb{R}^+, H^s(\Gamma))$, $\sigma > 0$:
$H^r_\sigma(\mathbb{R}^+, H^s(\mathbb{R}^n))$ defined using Fourier–Laplace transform

$$\left\{\psi : \text{supp } \psi \subset \overline{\mathbb{R}_+} \times \mathbb{R}^n,\ \int_{\mathbb{R}+i\sigma} d\omega \int_{\mathbb{R}^n} d\xi |\omega|^{2r}(|\omega|^2 + |\xi|^2)^s |\mathcal{F}\psi(\omega,\xi)|^2 < \propto \right.$$

$\sigma \rightsquigarrow$ equivalent to weight $\mathrm{e}^{-2\sigma t}dt$
recently rediscovered/used for FEM by Chaumont-Frelet, Ern et al.

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Space-time variational formulation of Dirichlet problem:
Find $\phi \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that $\forall \psi \in H^1_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$:

$$\langle \mathcal{V}\partial_t \phi, \psi\rangle = \langle \partial_t f, \Psi\rangle$$

The solution ϕ exists for $f \in H^2_\sigma(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma))$.

Key: $\mathcal{V}\partial_t$ coercive with loss (for wave eq.: Bamberger – Ha Duong '86)

$$\|\phi\|^2_{1,-\frac{1}{2},\Gamma} \gtrsim_\sigma \langle \mathcal{V}\partial_t\phi, \phi\rangle \gtrsim_\sigma \|\phi\|^2_{0,-\frac{1}{2},\Gamma}$$

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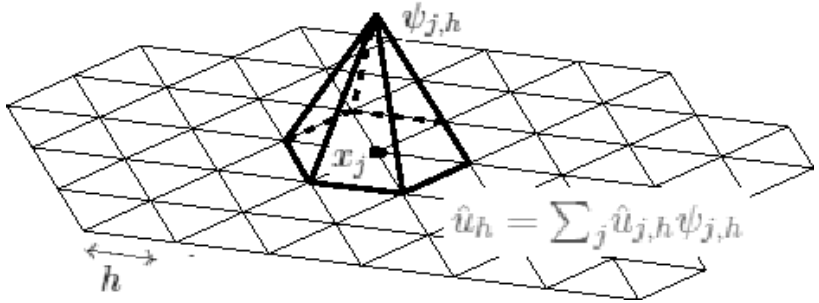
$$\|\phi\|^2_{1,-\frac{1}{2},\Gamma} \gtrsim \langle \mathcal{V}\partial_t \phi, \phi\rangle \gtrsim \|\phi\|^2_{0,-\frac{1}{2},\Gamma}$$

Loss avoided for different function spaces / bilinear forms by Steinbach, Urzua-Torres (SIMA '22) and subsequent works.

Discretization

- $\Gamma = \cup_{i=1}^M \Gamma_i$ triangulation
- V_h^p piecewise polynomial functions of degree p on $\Gamma = \cup_{i=1}^M \Gamma_i$ (continuous if $p \geq 1$)
- $[0,T) = \cup_{n=1}^L [t_{n-1}, t_n)$, $t_n = n(\Delta t)$
- $V_{\Delta t}^q$ piecewise polynomial functions of degree q in time (continuous and vanishing at $t = 0$ if $q \geq 1$)



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Time domain BEM: Find $\phi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$ such that $\forall \psi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$:

$$\langle \mathcal{V} \partial_t \phi_{h,\Delta t}, \psi_{h,\Delta t} \rangle = \langle \partial_t f, \psi_{h,\Delta t} \rangle$$

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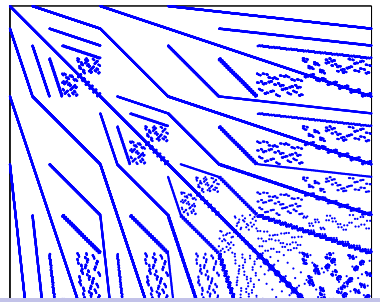
Simplest case: tensor products in space-time $V_{h,\Delta t}^{p,q} = V_h^p \otimes V_{\Delta t}^q$
causality $\rightsquigarrow$ block triangular Galerkin matrix $\rightsquigarrow$ backsubstitution:
compute 1 matrix per time step



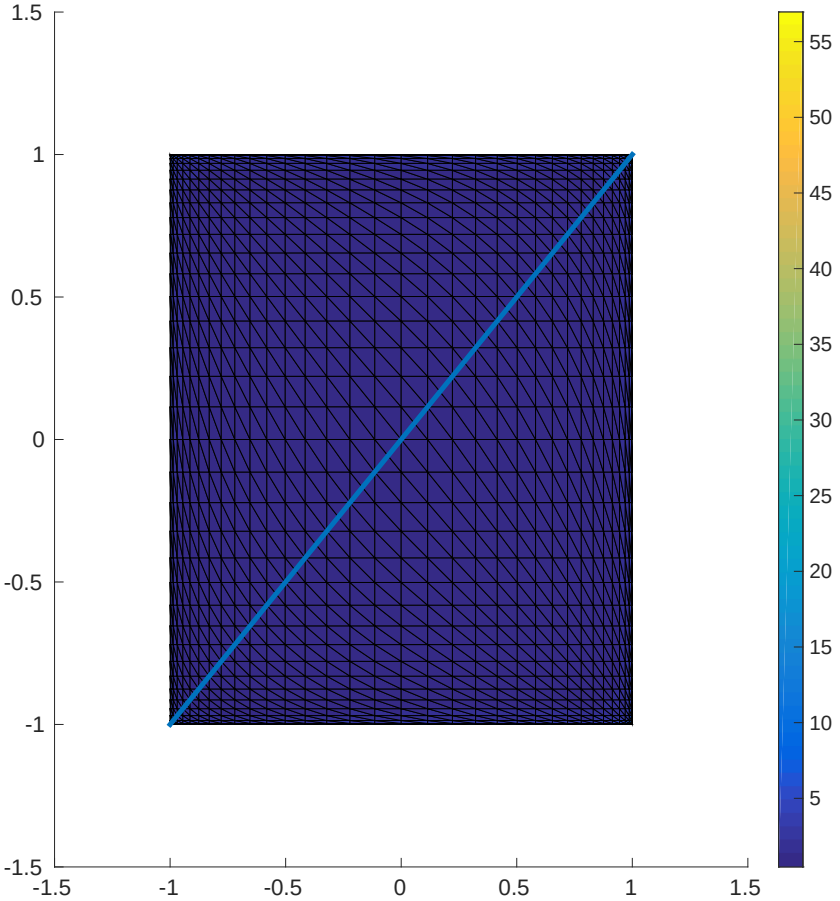
$$\forall n: \sum_{m=1}^{n} V^{n-m} \Phi^m = F^n$$

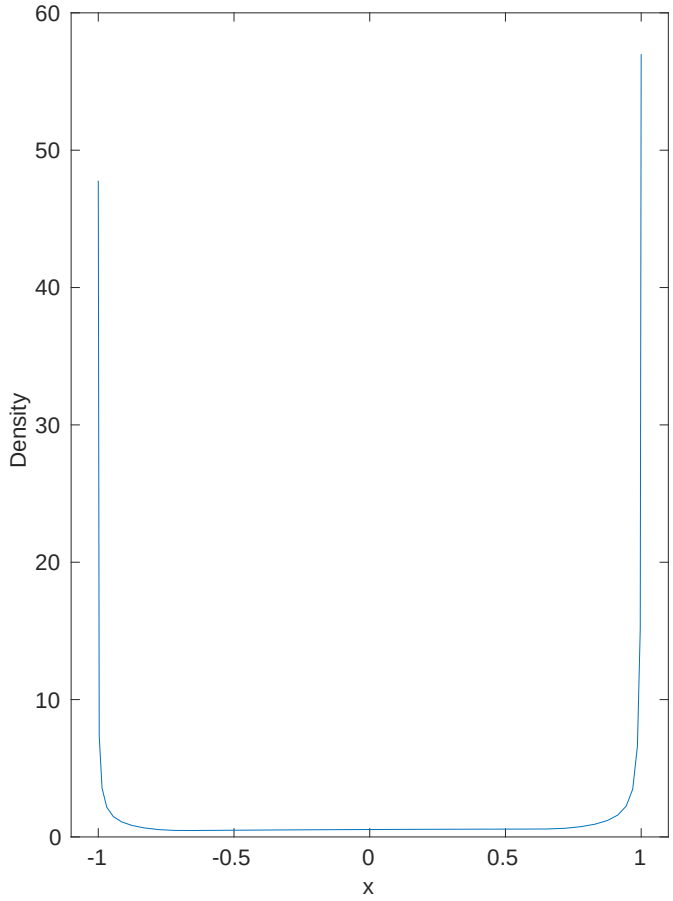
$$\Longleftrightarrow \quad V^0 \Phi^n = F^n - \sum^{n-1} V^{n-m} \Phi^m$$

Screen problems

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$.

solution near corner $r^{-0.703\ldots}$, near edge $r^{-\frac{1}{2}}$





Singularities at edges and corners

The next theorem in $2d$ goes back to Plamenevskii (a), Müller–Schwab (b) for FEM, 2d/3d HG, Özdemir, Stark, Stephan.

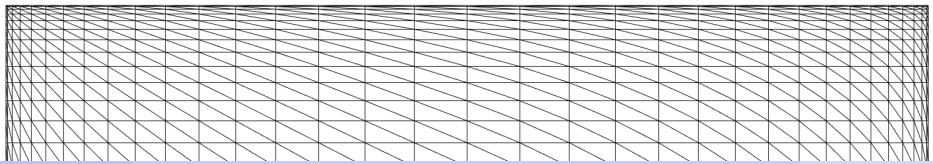
Theorem

a) In $2d$, solution behaves like $r^{-\frac{1}{2}}$ at $\partial\Gamma$.
In $3d$ $r^{\gamma-1}$ near corner, $r^{-\frac{1}{2}}$ at $\partial\Gamma$.

b) Optimal approximation on β-graded mesh in energy norm ($\Delta t \leq h$):
Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \lesssim h^{\min\{\frac{\beta}{2},\frac{3}{2}\}-\varepsilon}$.

Extensions: elastodynamics, polygonal domains, Neumann problem (Aimi, Di Credico, HG, Stephan, Numer. Math. '23).

$$x_j = 1 - \left(\frac{j}{N}\right)^\beta, \ j = 1, \dots, N \ .$$

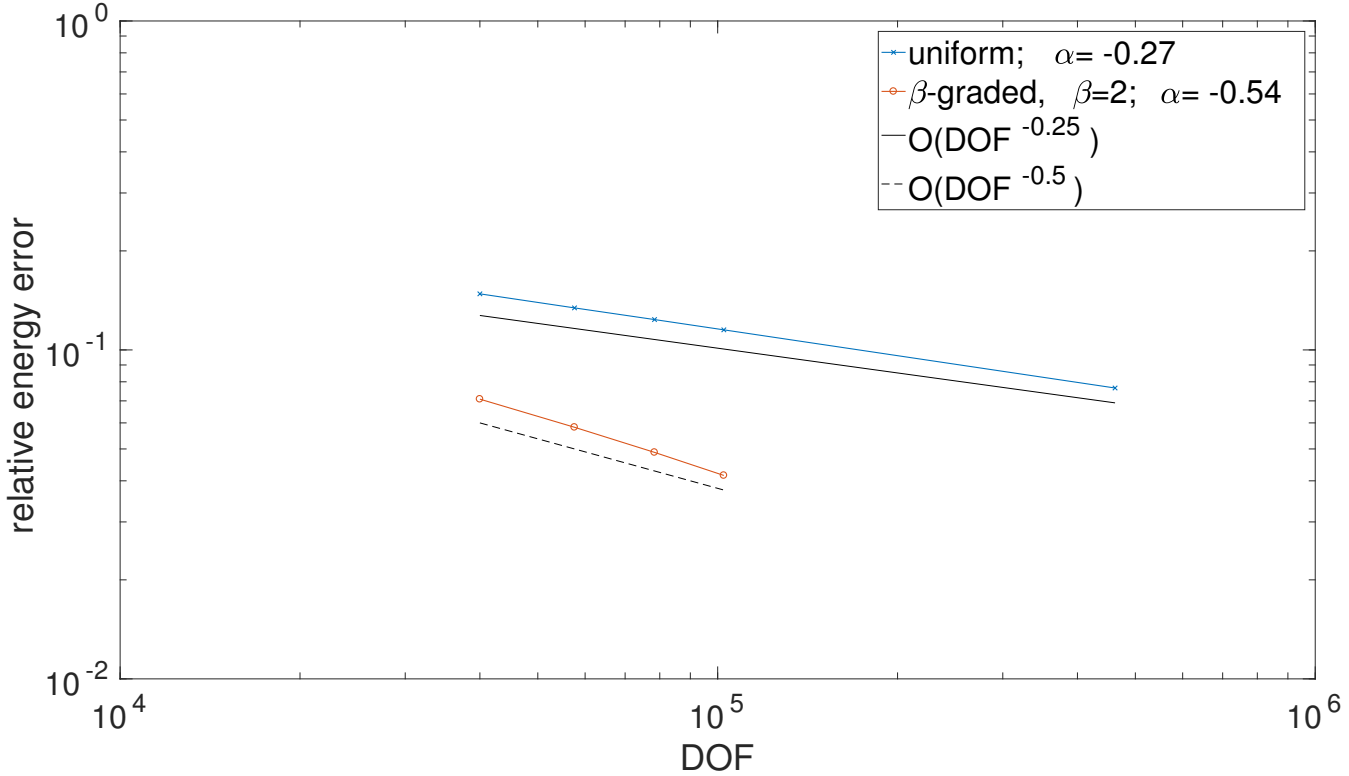


Screen problems: convergence rates for $3d$ wave equation

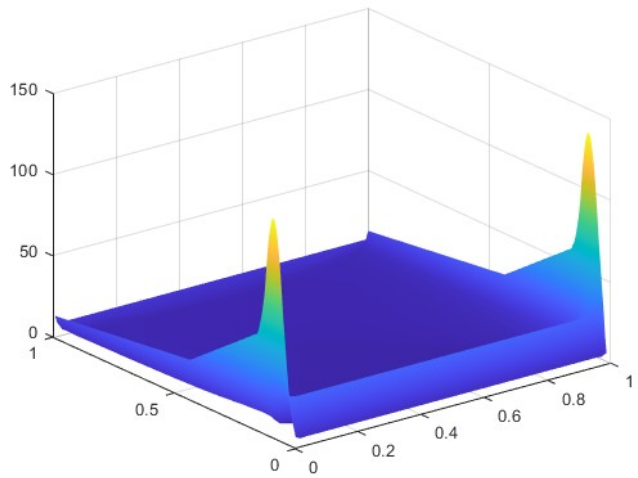
$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 1$.

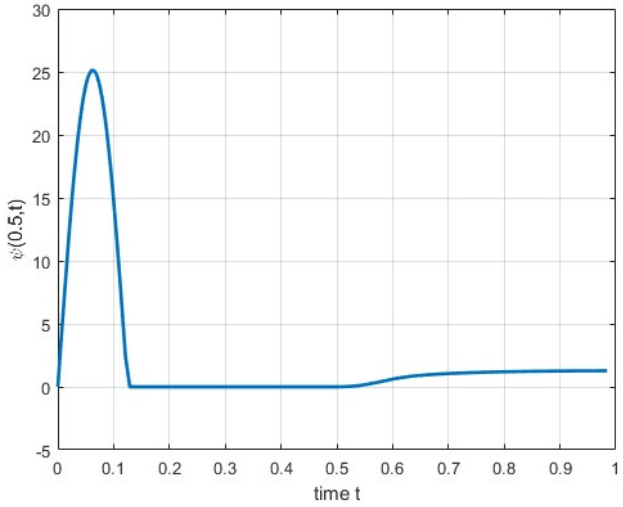
Energy norm^2 $= \langle \mathcal{V}\partial_t(\phi_h - \phi), \phi_h - \phi\rangle \sim h^2 \simeq DOF(\Gamma)^{-1}$ (2-graded)

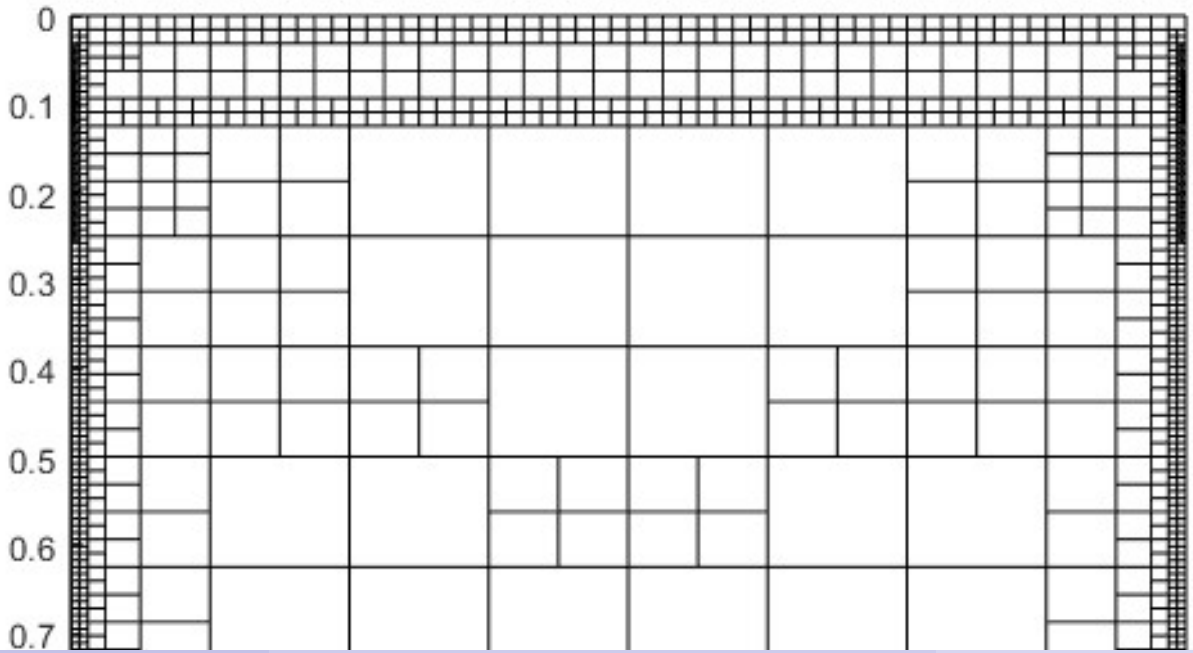
$\sim h \simeq DOF(\Gamma)^{-1/2}$ (uniform)



Goal: Space-time adaptive mesh refinements







A posteriori error estimate for $\mathcal{V}\phi = f$

Theorem (HG, Özdemir, Stark, Stephan, Numer. Math. '20)

Let $\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that
$\mathcal{R} = \partial_t\, g - \mathcal{V}\partial_t\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^1(\Gamma)) \Longrightarrow$

$$\|\phi - \phi_{h,\Delta t}\|^2_{0,-\frac{1}{2},*} \lesssim \sum_{\square=[t_i,t_{i+1}]\times\Delta} \operatorname{diam}_\square \|\mathcal{R}\|^2_{0,1,\square}$$

$$\operatorname{diam}_\square \|\mathcal{R}\|^2_{0,1-\epsilon} \lesssim \|\phi - \phi_{h,\Delta t}\|^2_{2,-\frac{1}{2},*}$$

The upper bound follows from a function-space argument (localization of fractional Sobolev norms, Carstensen '96), for large classes of meshes.

The lower bound holds on quasi-uniform meshes.
(Even for time-independent problems little is known about lower bounds.)

The lower bound is weaker than the energy norm ($\hookleftarrow$ loss of derivatives).

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Extension to $\mathcal{W}$ with Aimi, Di Credico, Guardasoni (CAMWA '25).

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For the time-continuous problem and smooth f, the norm $\|\phi - \phi_h\|^2_{0,-\frac{1}{2},*}$ on the left hand side can be replaced by the optimal $\|\phi - \phi_h\|^2_{1,-\frac{1}{2},*}$.
(Chaumont-Frelet, SISC '23 / Chaumont-Frelet, HG)

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Residual error indicators:

$$\eta^2(\square) = \operatorname{diam}_\square \, \left(\|\partial_t\mathcal{R}\|^2_{0,0,\square} + \|\nabla_\Gamma\mathcal{R}\|^2_{0,0,\square}\right)$$

$$\tilde\eta^2(\square) = \|\partial_t\mathcal{R}\|^2_{0,0,\square} + h_\square \, \|\nabla_\Gamma\mathcal{R}\|^2_{0,0,\square}$$

(Simple) Proof of upper bound

$$\|\phi - \phi_{h,\Delta t}\|^2_{H^{0,-\frac{1}{2}}([0,T],\Gamma)}$$

$$\begin{aligned}
&\lesssim \int_0^T dt \int_0^t ds \int_\Gamma d\Gamma\ \mathcal{V}(\dot\phi - \dot\phi_{h,\Delta t})(\phi - \phi_{h,\Delta t}) \\
&= \int_0^T dt \int_0^t ds \int_\Gamma d\Gamma\ (\dot f - \mathcal{V}\dot\phi_{h,\Delta t})(\phi - \phi_{h,\Delta t}) \\
&\lesssim_T \ \|\mathcal{R}\|_{H^{0,\frac{1}{2}}([0,T],\Gamma)}\ \|\phi - \phi_{h,\Delta t}\|_{H^{0,-\frac{1}{2}}([0,T],\Gamma)}\ .
\end{aligned}$$

- interpolation inequality:

$$\|\mathcal{R}\|^2_{H^{0,\frac{1}{2}}} \lesssim \|\mathcal{R}\|_{H^{0,1}} \|\mathcal{R}\|_{L^2L^2}\ .$$

- residual orthogonal: $\mathcal{R} \perp \psi_{h,\Delta t}$.
- interpolation $\rightsquigarrow$ $h, \Delta t$.

Lower bound for Dirichlet bvp in $\mathbb{R}^3$

dumb estimate:

$$\|\mathcal{R}\|_{H^{r-1,s+\frac{1}{2}}} = \|\mathcal{V}(\dot\phi - \dot\phi_{h,\Delta t})\|_{H^{r-1,s+\frac{1}{2}}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{r+1,s-\frac{1}{2}}} \ .$$

$$\|\mathcal{R}\|_{H^{0,1-\varepsilon}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{2,-\varepsilon}} \ .$$

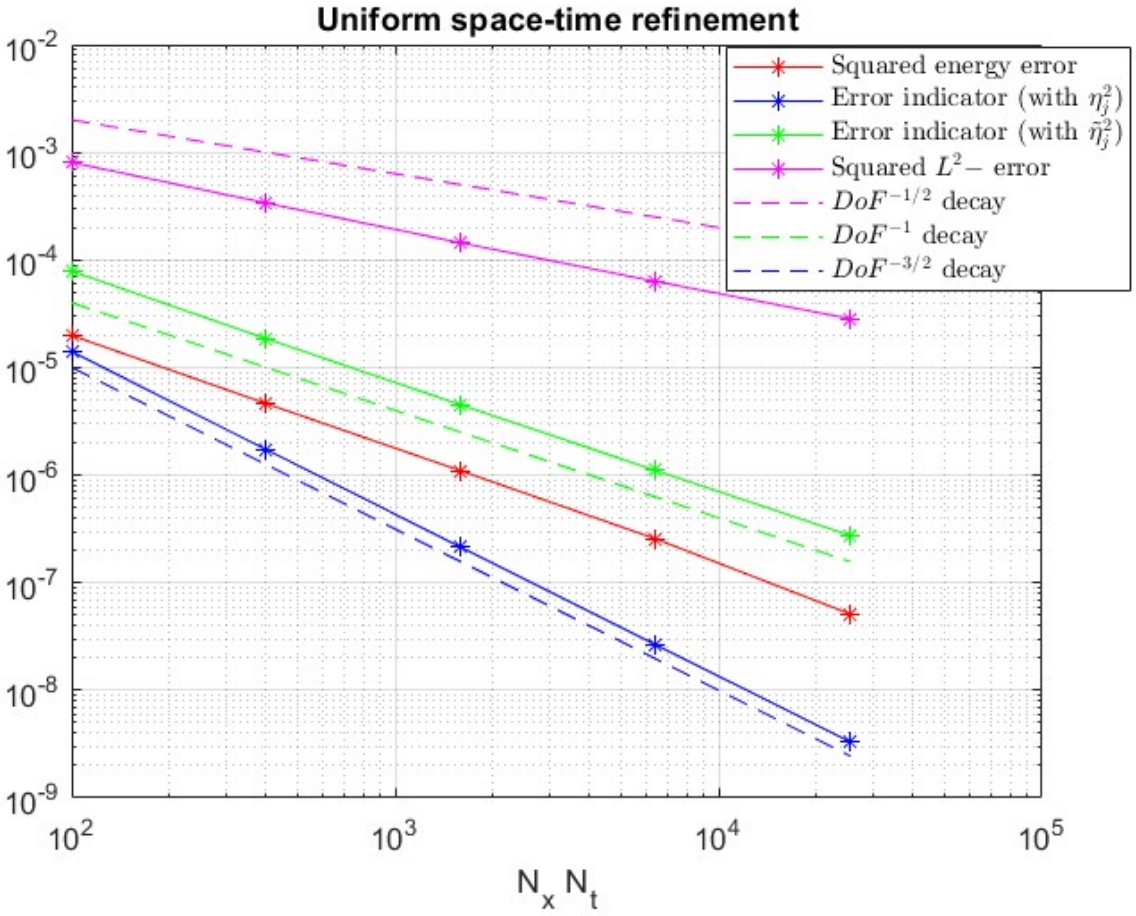
Using the singular expansion for $\partial\Gamma \neq \emptyset$, we estimate $\|\phi - \phi_{h,\Delta t}\|_{H^{2,-\varepsilon}}$ on quasi-uniform meshes to obtain the efficiency of the estimator.

- approximation: $\frac{\|\phi-\phi_{h,\Delta t}\|_{H^{-1}(\Gamma)}}{\|\phi-\phi_{h,\Delta t}\|_{L^2(\Gamma)}}$ ($\leftarrow\!\!\!\rightarrow$ problem: no elliptic regularity)
- Lack of elliptic regularity: Error of best approximation? ($\leftarrow\!\!\!\rightarrow$ use singular expansion to estimate on quasi-uniform meshes)
- inverse estimates
- stable interpolation operators, etc.

Error indicator $\sim$ energy error on uniform mesh

2D: $\phi(t,x) = tx$ on $\Gamma = [0,1] \times \{0\}$, $0 < t < 1$.

We consider the error indicators on a sequence of uniform meshes.
Compare to error in energy norm.



A posteriori error estimate for $\mathcal{V}\phi = f$

Theorem (Chaumont-Frelet, HG, Labarca-Figueroa, Nick, '25)

The discretization error of the full CQ discretization (with BDF-1 or BDF-2) is bounded, for all $r \in \mathbb{N}$, $\omega > 0$, by

$$\|\varphi - \varphi_h^\tau\|_{L^2(0,T;\widetilde{H}^{-1/2}(\Gamma))} \leq C\big(1+\omega^2\big)\,\|f - V(\partial_t^\tau)\varphi_h^\tau\|_{L^2(0,T;H^{1/2}(\Gamma))} + \frac{C}{\omega^r}\,\|f\|_{H_0^{r+6}(0,T;H^{1/2}(\Gamma))} + C\tau^{2m-1}.$$

Instead of ω, the estimate can be written with 2nd time-derivatives on rhs.

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$$\|\varphi - \varphi_h^\tau\|_{L^2(0,T;\widetilde{H}^{-1/2}(\Gamma))} \leq C\big(1+\omega^2\big)\,\|f - V(\partial_t^\tau)\varphi_h^\tau\|_{L^2(0,T;H^{1/2}(\Gamma))} + \frac{C}{\omega^r}\,\|f\|_{H_0^{r+6}(0,T;H^{1/2}(\Gamma))} + C\tau^{2m-1}.$$

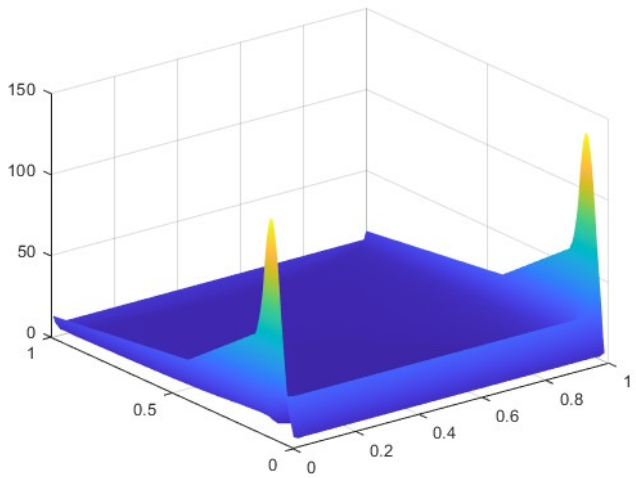
Localization (2d):

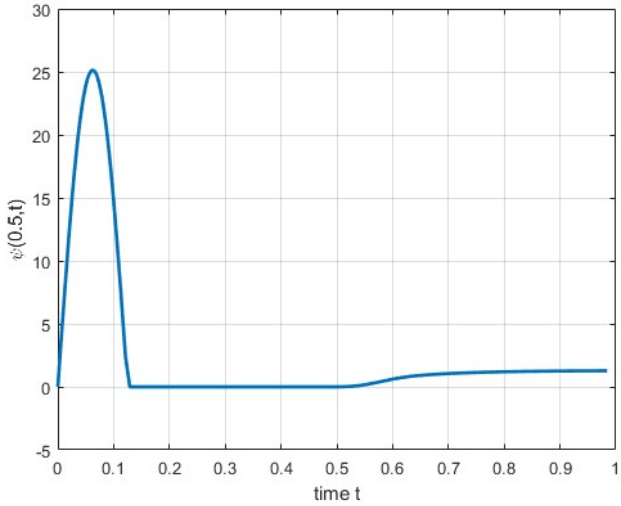
$$\|f - V(\partial_t^\tau)\varphi_h^\tau\|^2_{l^2(0,T;H^{1/2}(\Gamma))} \leq \widetilde{C}_\Gamma \tau \sum_{t_i}\sum_{E_j} h_{E_j} \int_{E_j} \left(\nabla_\Gamma (f(t_i,x) - V(\partial_t^\tau)\varphi_h^\tau(t_i,x))\right)^2 dx,$$

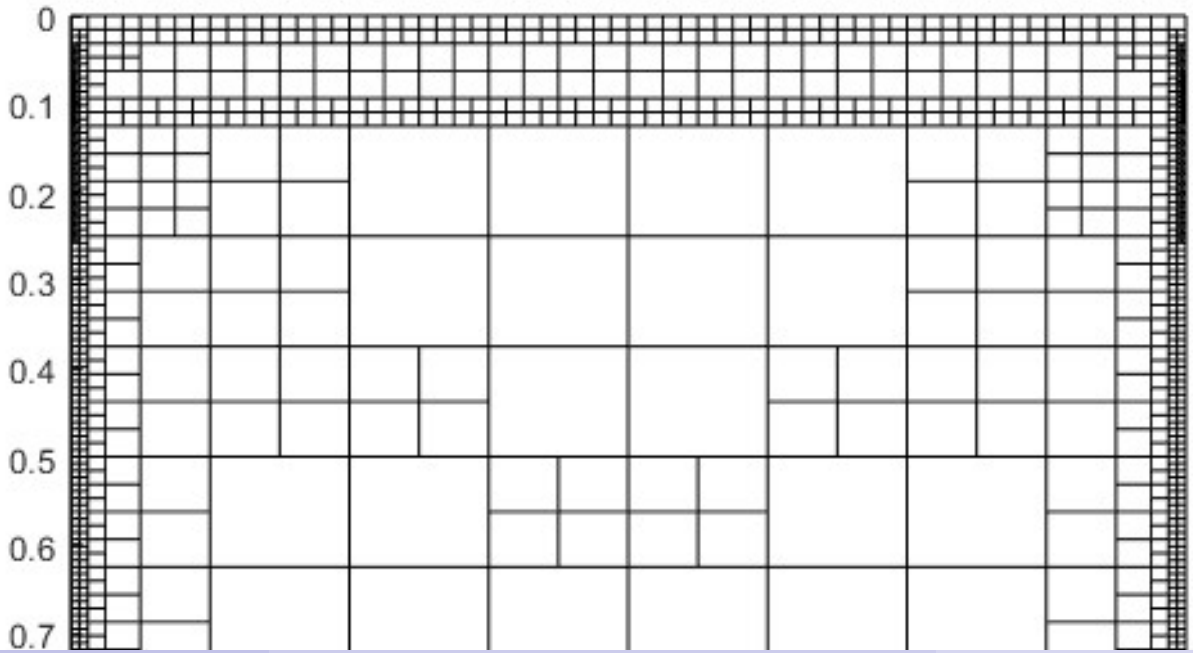
Residual error indicators:

$$\eta(E_j)^2 = \tau h_{E_j} \sum_{t_i} \int_{E_j} \left(\nabla_\Gamma (f(t_i,x) - V(\partial_t^\tau)\varphi_h^\tau(t_i,x))\right)^2 dx$$

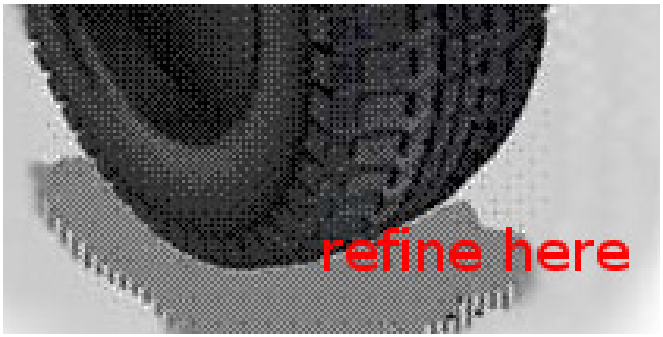
Goal: Space-time adaptive mesh refinements







A first adaptive method: space-adaptivity

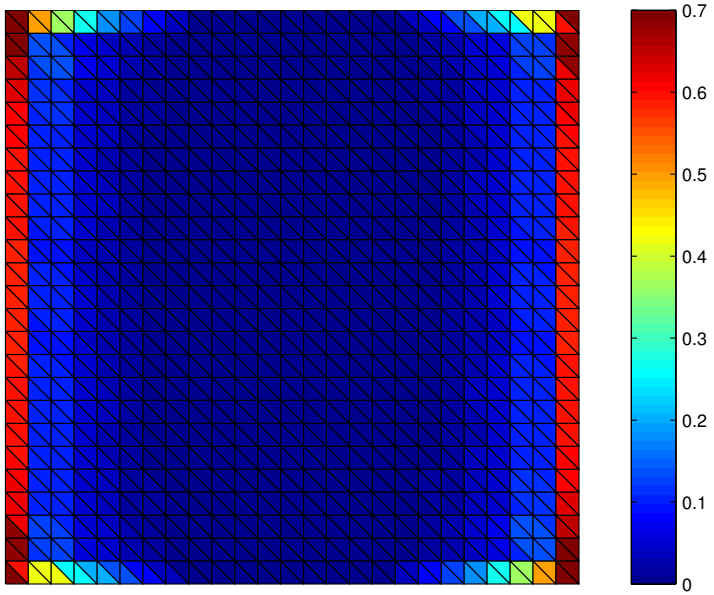


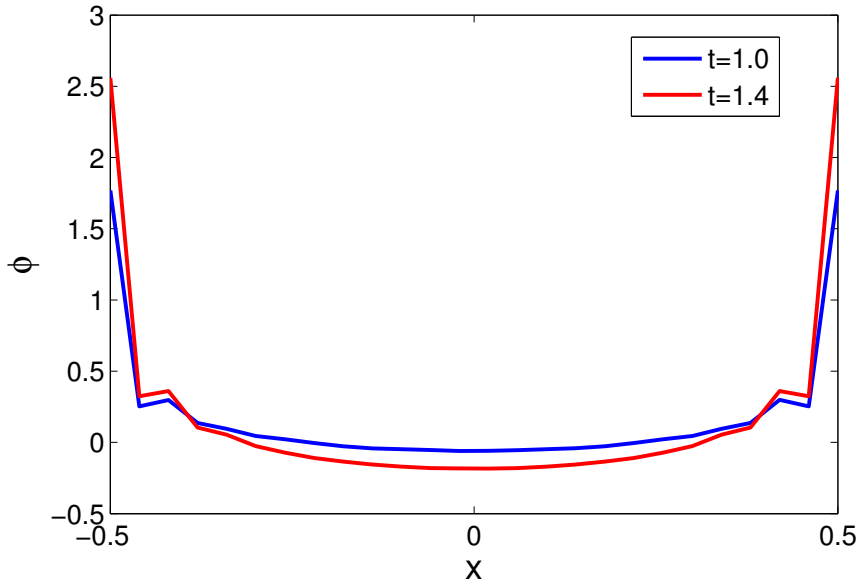
1. Start with coarse space-time grid: $(\Delta t) \simeq (\Delta x) \simeq h_0 \ \forall \Delta, i$
2. Solve discretisation of $\mathcal{V}\dot{\phi} = \dot{g}$.
3. Compute time-integrated error indicator $\eta^2(\Delta) = \sum_i \eta^2(\Delta, i)$
4. $\sum_\Delta \eta^2(\Delta) < \varepsilon \quad \Longrightarrow \quad$ STOP
5. $\eta(\Delta) > \delta \eta_{max} \quad \Longrightarrow \quad \Delta \to \Delta/4, \ (\Delta t) \to \frac{(\Delta t)}{2}$
6. GO TO 2.

Space-adaptive refinements on screen

3D: $\mathcal{V}\phi = \sin^5(t)x^2$ on $\Gamma = [-0.5, 0.5]^2 \times \{z = 0\}$, $0 < t < 2.5$, $\Delta t = 0.1$.

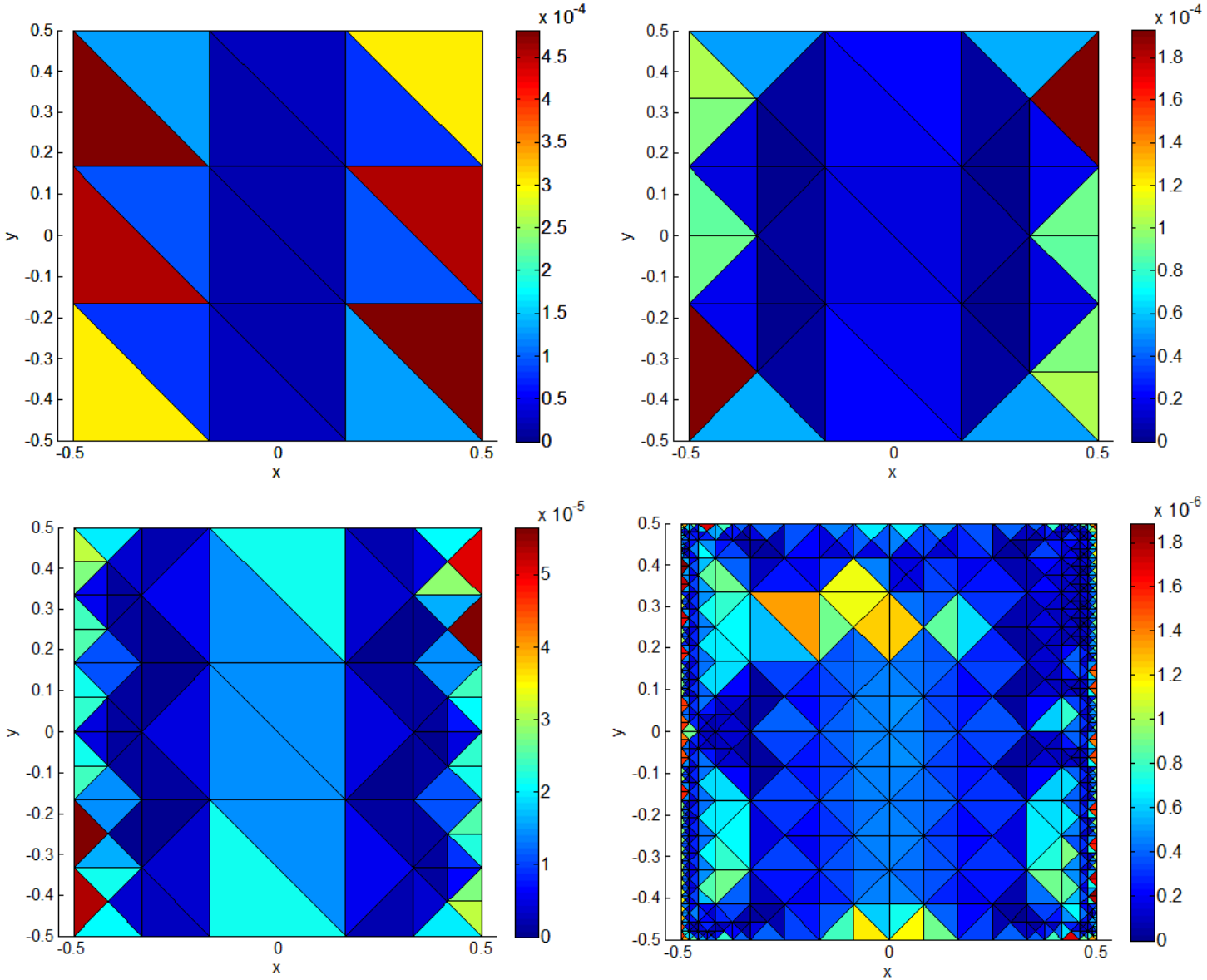
Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.





- Uniform method: Density ϕ at $t = 1.0, 1.4$

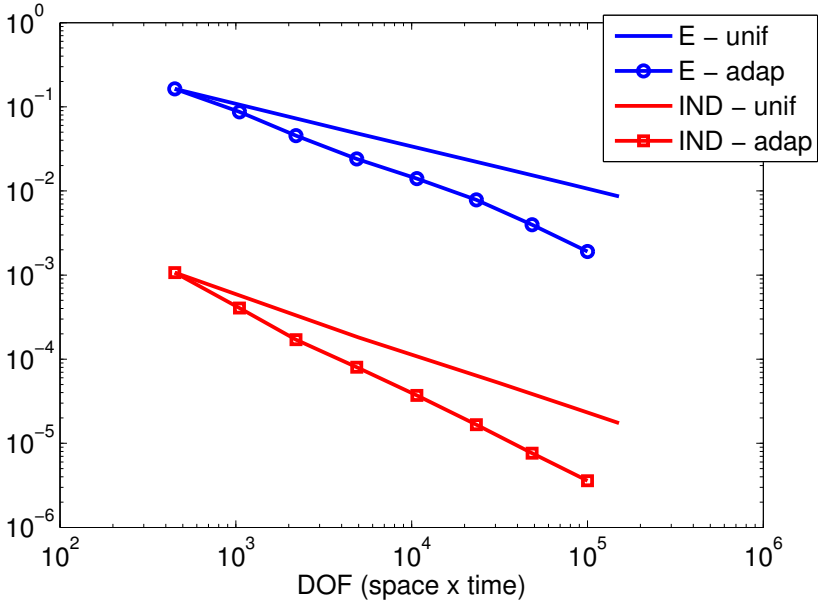
Space-adaptive refinements on screen: meshes

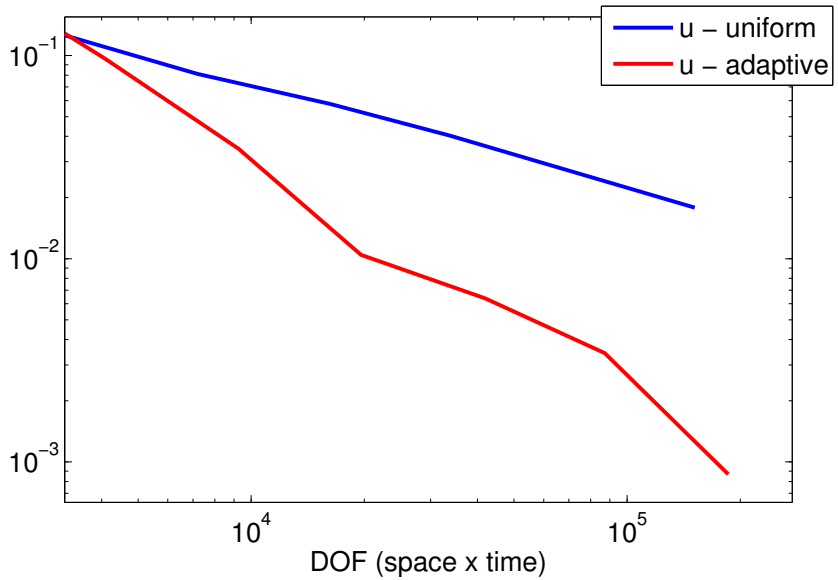


Space-adaptive refinements on screen: convergence

3D: $\mathcal{V}\phi = \sin^5(t)x^2$ on $\Gamma = [-0.5, 0.5]^2 \times \{z = 0\}$, $0 < t < 2.5$, $\Delta t = 0.1$.

Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.





- Convergence rate 0.5 (uniform), 0.77 adaptive reproduces rates for time-independent BEM.

Space-adaptive refinements using CQ

2D: $\mathcal{V}\phi =$ incoming wave on trapping obstacle Γ, $0 < t < 10$, $\Delta t = 0.1$.

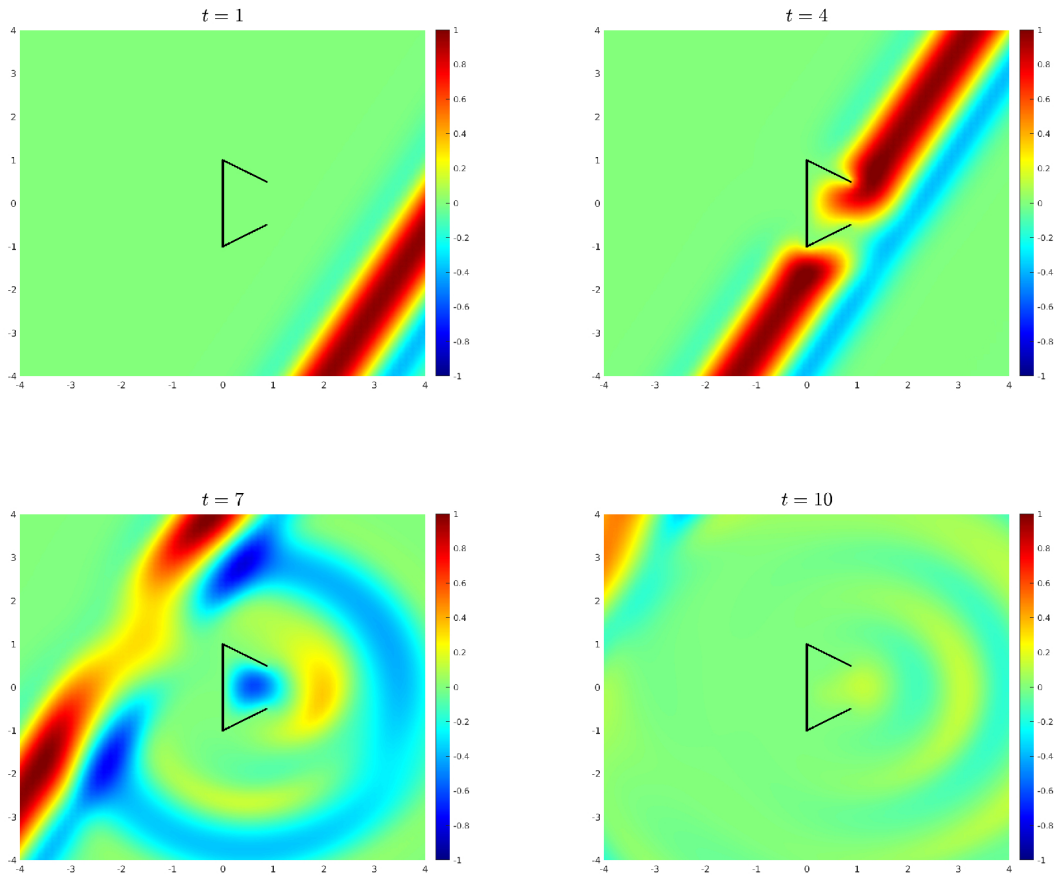
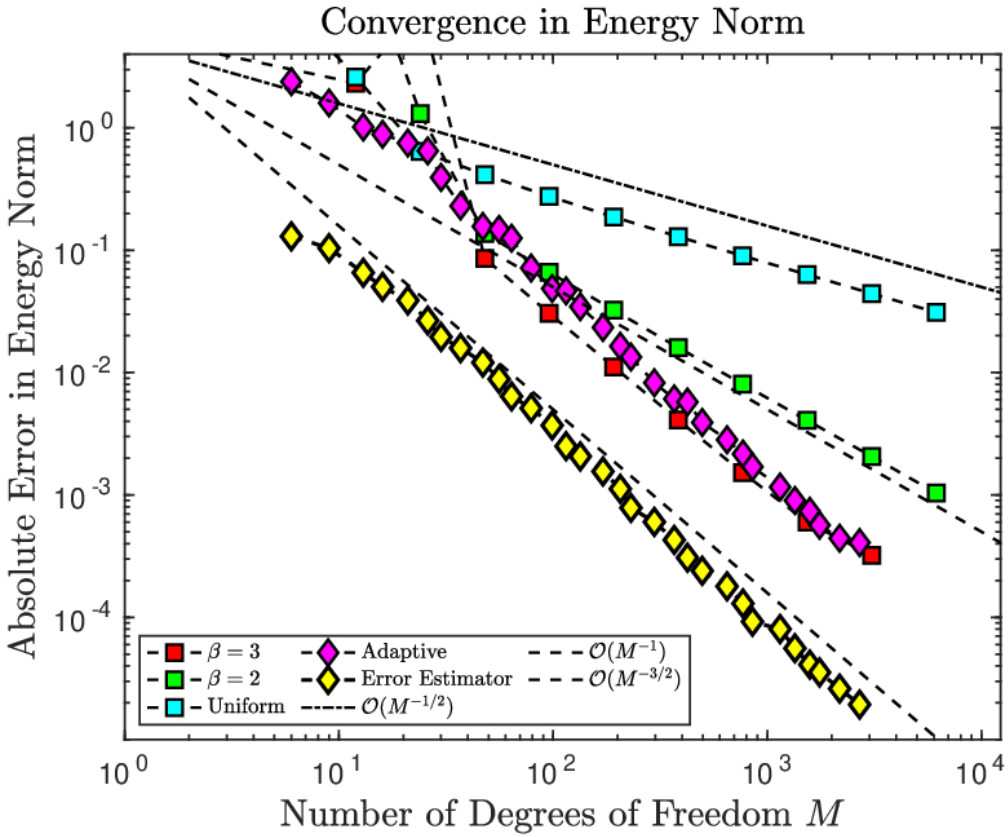


Figure: Scattering at a screen with trapping geometry: snapshots of solution u_h^τ

Chaumont-Frelet, HG, Labarca-Figueroa, Nick, preprint (2025)

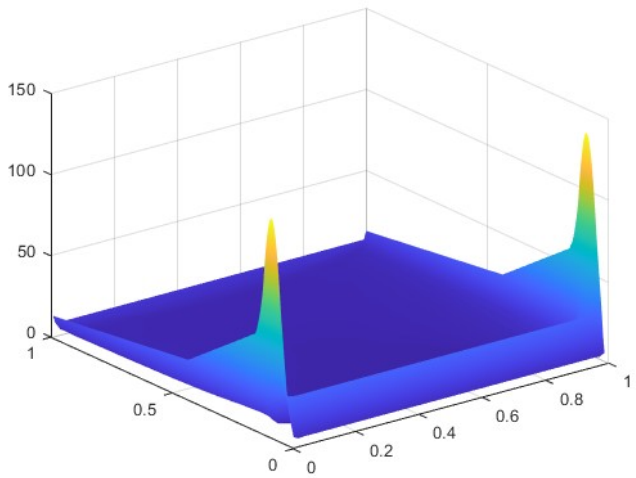
Space-adaptive refinements using CQ

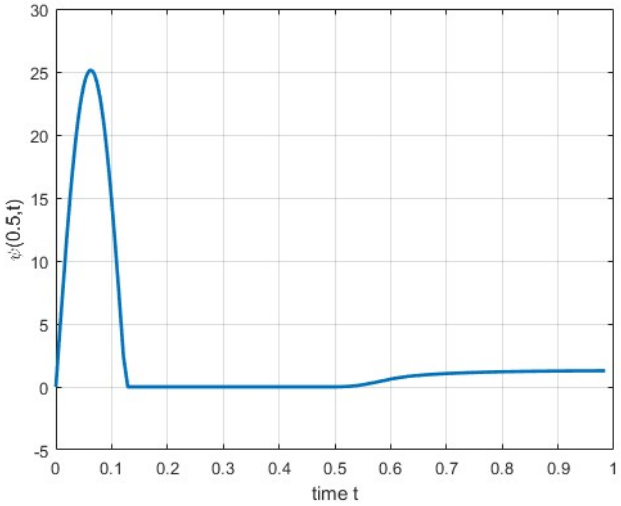
2D: $\mathcal{V}\phi =$ incoming wave on trapping obstacle Γ, $0 < t < 10$, $\Delta t = 0.1$.

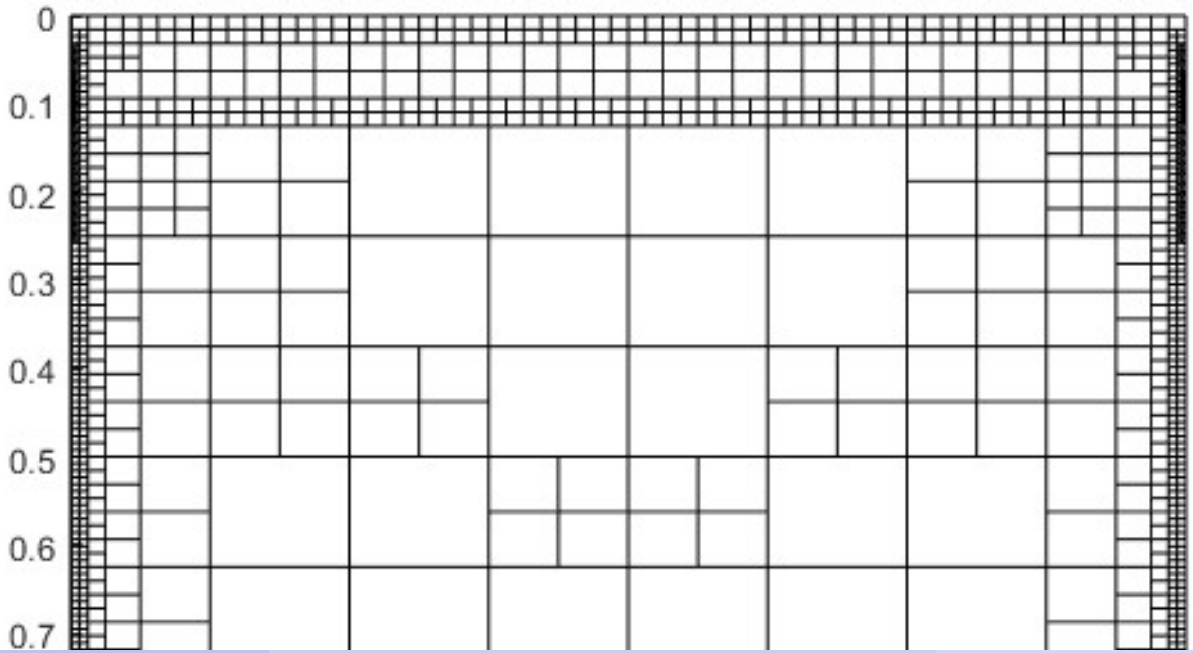


Chaumont-Frelet, HG, Labarca-Figueroa, Nick, preprint (2025)

Goal: Space-time adaptive mesh refinements







Space-time adaptive mesh refinements

1. Start with coarse space-time grid: $(\Delta t) \simeq (\Delta x) \simeq h_0$ for all elements
2. Solve discretisation of $\mathcal{V}\dot{\phi} = \dot{g}$.
3. Compute error indicator $\eta((t_i, t_{i+1}] \times \Delta)$
4. $\sum_{i,\Delta} \eta^2((t_i, t_{i+1}] \times \Delta) < \varepsilon \quad \Longrightarrow \quad$ STOP
5. $\eta((t_i, t_{i+1}] \times \Delta) > \delta \eta_{max} \quad \Longrightarrow \quad (\Delta t)_i \to \frac{(\Delta t)_i}{2}$, $\Delta \to \Delta/4$
6. GO TO 2.

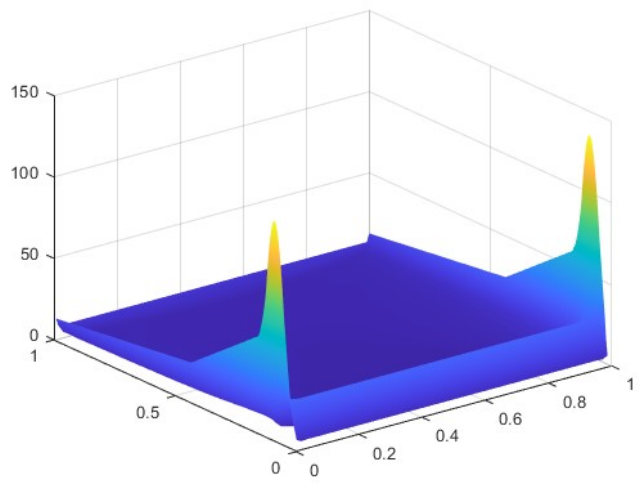
Algorithmic challenges:

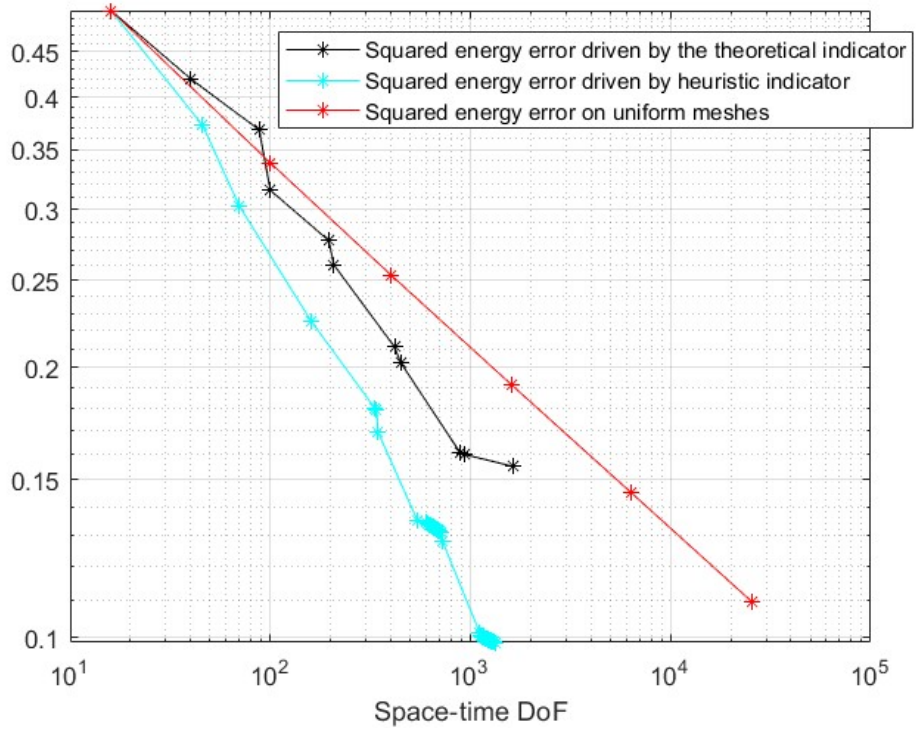
- Matrix assembly for non-product meshes (= new code)
- Loss of block-Toeplitz structure:
  assemble & store full space-time matrix (= reuse information)

Efficient for solutions with time-dependent singularities

Space-time adaptive mesh refinements

2D: $\mathcal{V}\phi = f$, $f(x,t) = t^{2/3}$ on $\Gamma = (0,1) \times \{0\}$, $0 < t < 1$.





Space-time adaptive mesh refinements

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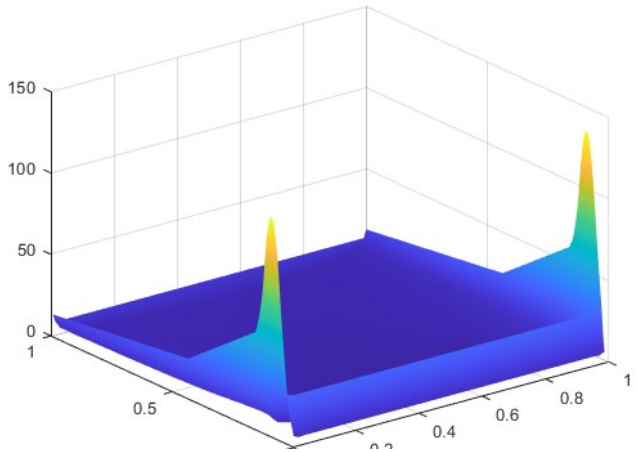
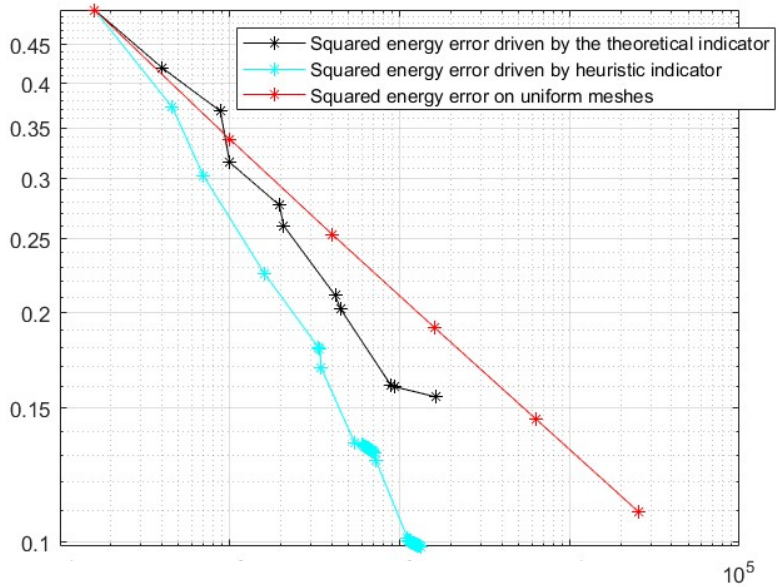
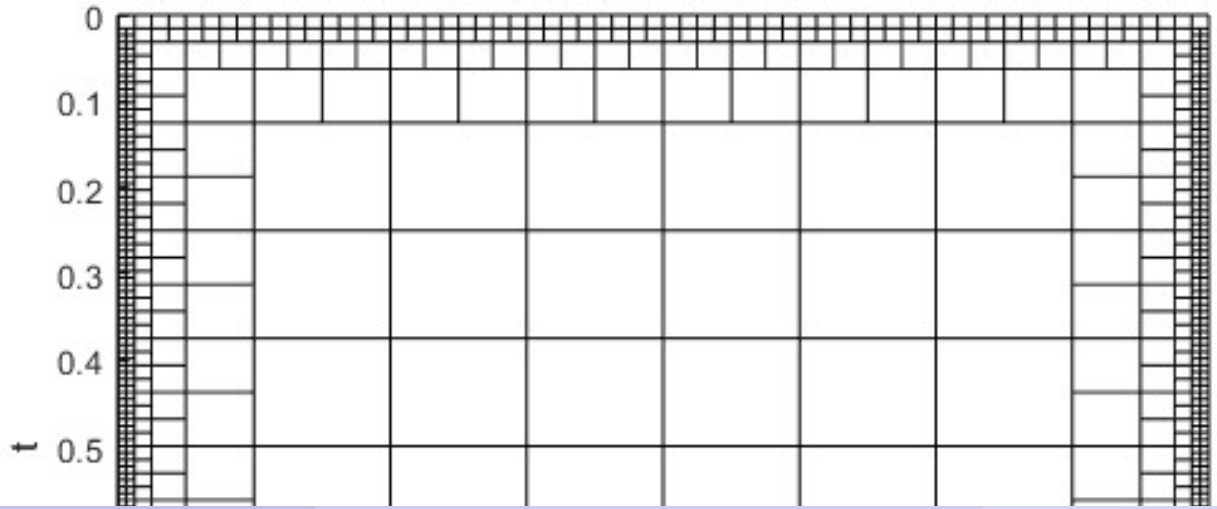
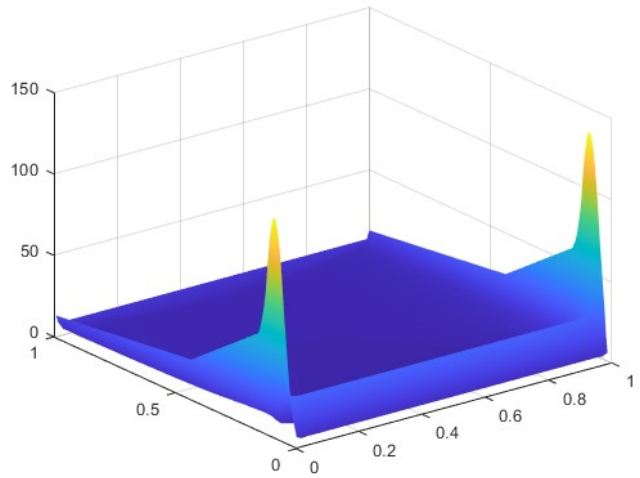
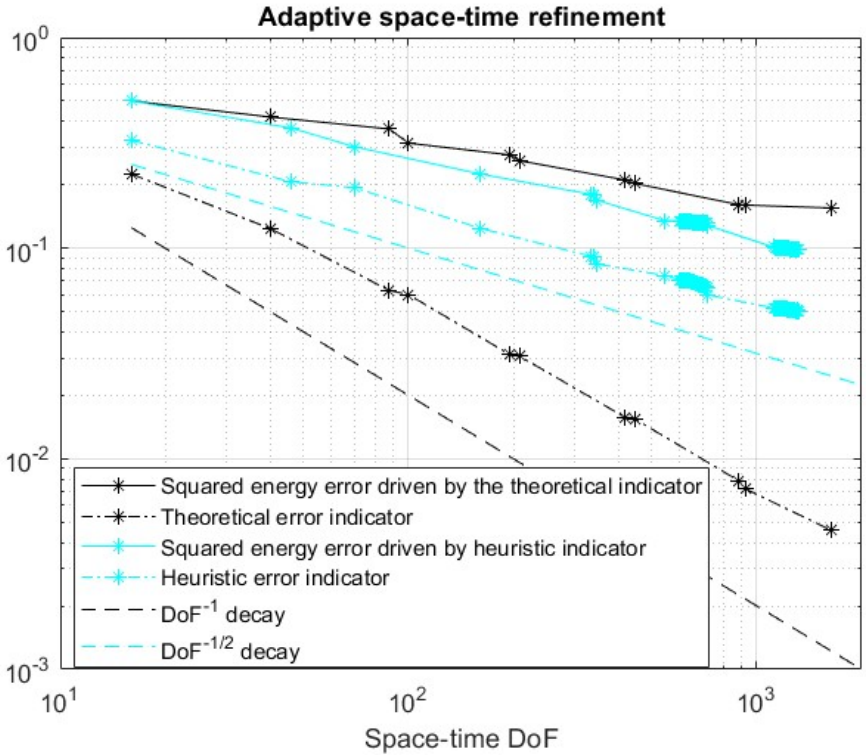






Space-time adaptive mesh refinements



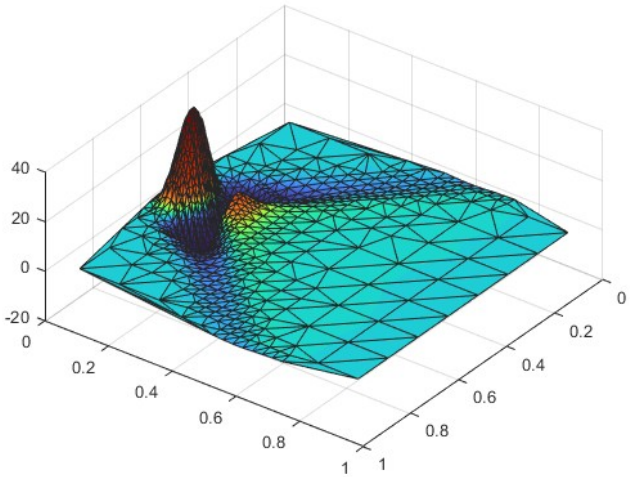


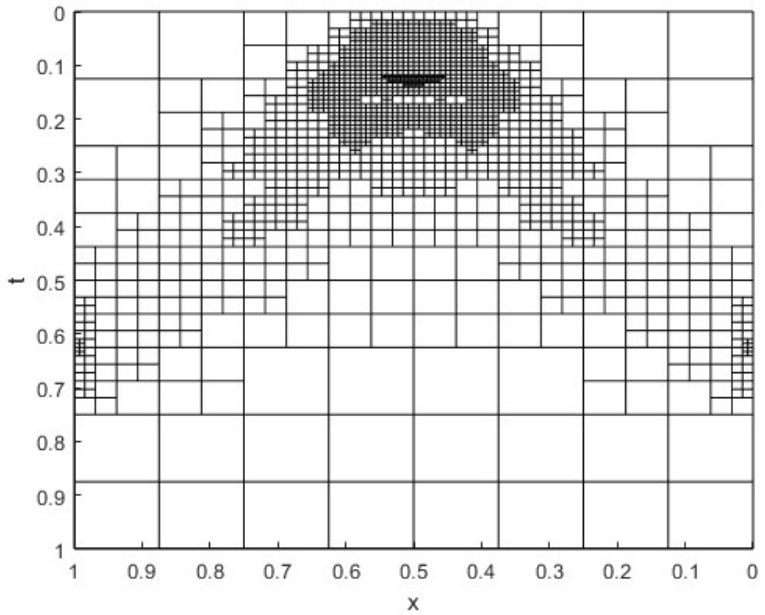
Space-time adaptive mesh refinements

2D: $\mathcal{V}\phi = f$, $f(x,t) =$

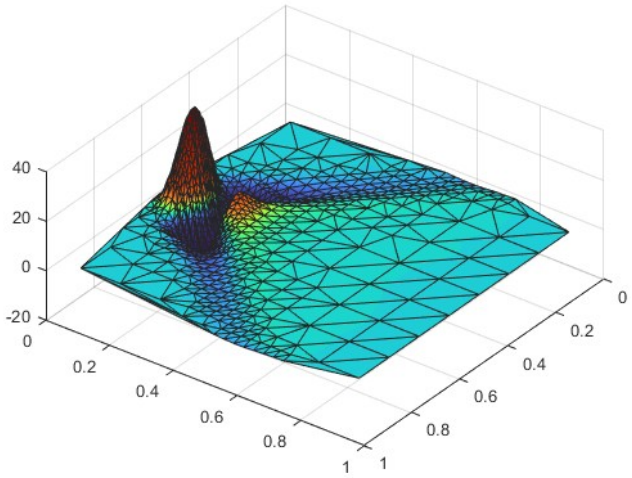
$$\begin{cases} \sin^4(4\pi t)\sin^4\big(3\pi(x-1)\big)H(x-\frac{1}{3})H(\frac{2}{3}-x) & \text{for} \quad 0 \le t \le \frac{1}{8} \\ \sin^4(4\pi(-t-\frac{2}{8})\sin^4\big(3\pi(x-1)\big)H(x-\frac{1}{3})H(\frac{2}{3}-x) & \text{for} \quad \frac{1}{8} \le t \le \frac{2}{8} \\ 0 & \text{for} \quad t \ge \frac{1}{8} \end{cases}$$

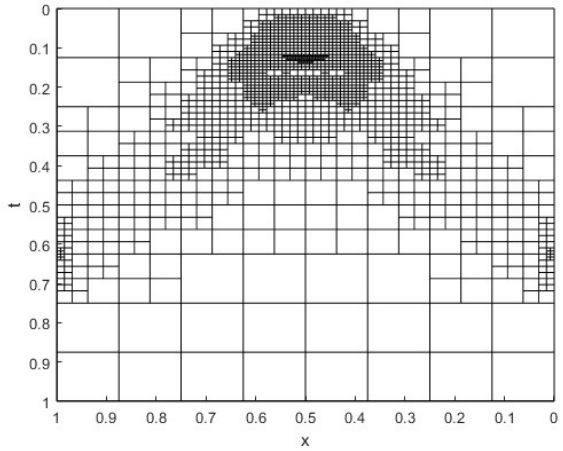
on $\Gamma = (0,1) \times \{0\}$ slit, $0 < t < 1$.

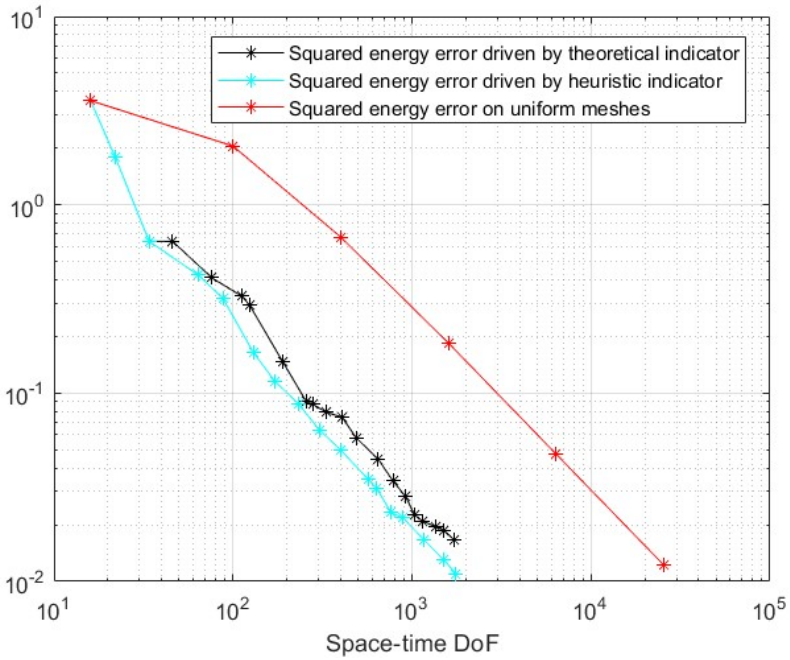


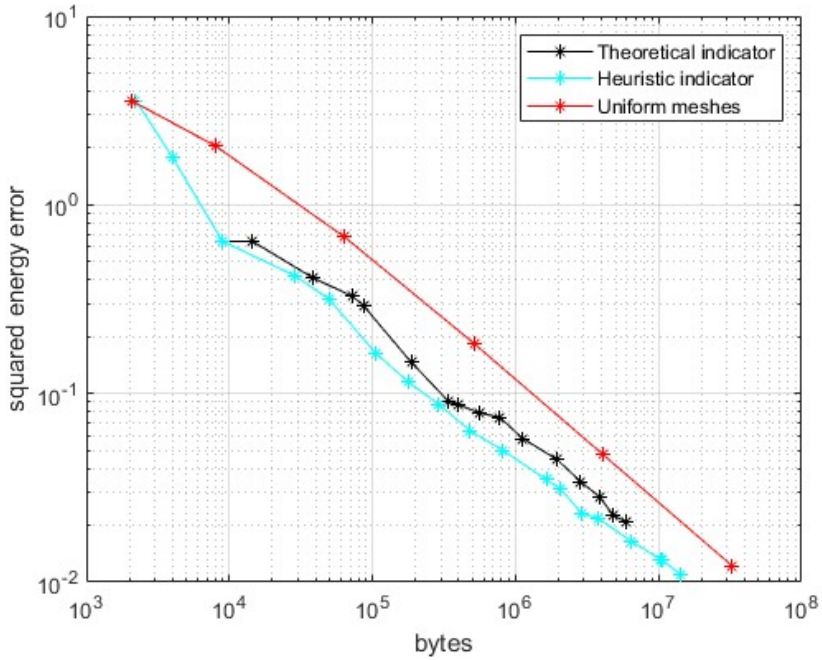


Space-time adaptive mesh refinements



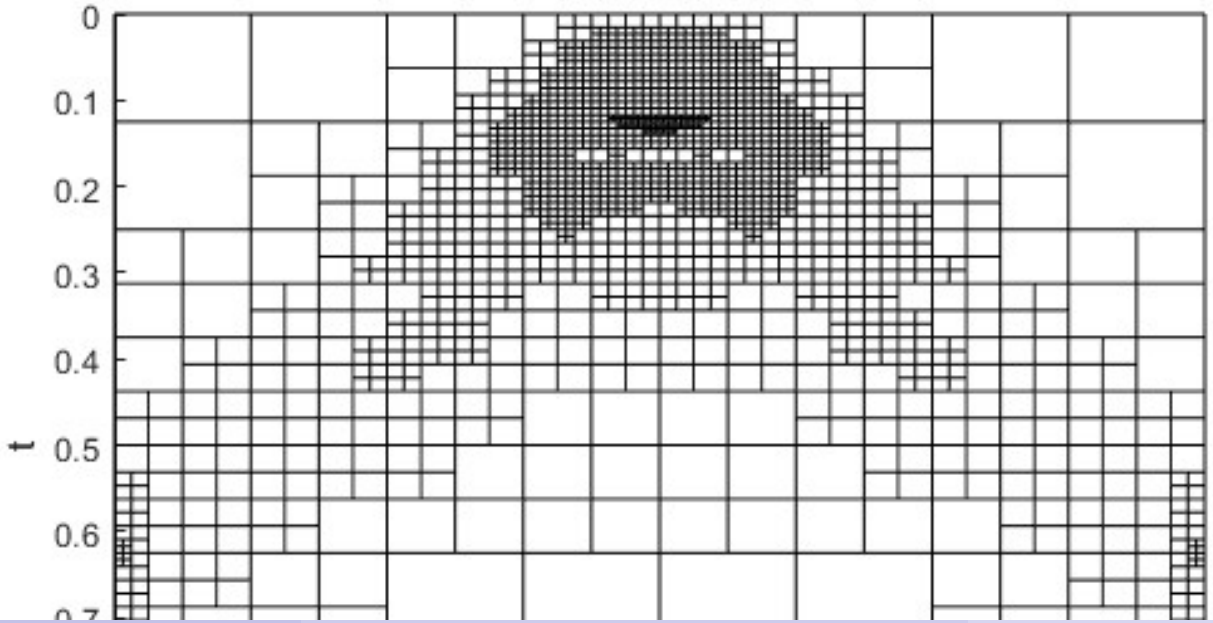






Conclusions: Time domain BEM + mesh refinements

- A posteriori analysis for elliptic BEM partly generalizes to space-time Galerkin and CQ methods, (currently) "loss" of time derivatives compared to elliptic case for data non-smooth in time
- Static meshes optimal for geometric singularities (edges, corners) $\rightsquigarrow$ space-only adaptive mesh refinements
- Space-time adaptive mesh refinements for Galerkin BEM.
  efficient (memory, time) for sufficiently strong singularities



Conclusions: Time domain BEM + mesh refinements

Outlook:

- (✓) Understand/remove "loss" between lower and upper bounds? (with T. Chaumont-Frelet)
- ✓ hypersingular operator: a posteriori estimates / adaptivity in space (with Aimi, Di Credico, Guardasoni, CAMWA '25)
- p-, hp-adaptivity?

adaptive convolution quadrature:
Chaumont-Frelet, HG, Labarca-Figueroa, Nick, *A posteriori error estimates and space-adaptive mesh refinements for time-dependent scattering problems*, arXiv:2509.04217.

space-time adaptivity:
Aimi, Di Credico, HG, Guardasoni, *A space-time adaptive boundary element method for the wave equation*, preprint (2025).

a posteriori estimate + space-only adaptivity:
HG, Özdemir, Stark, Stephan, Numer. Math. 146 (2020), 239-280.