The magnitude function: *Spectral geometry and fractional-order boundary problems*

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(joint with Magnus Goffeng and Nikoletta Louca)

Oldenburg Analysis Seminar, 3 June 2021

Examples of "size"

A good notion of size satisfies:

$$\mathrm{Size}(A \cup B) = \mathrm{Size}(A) + \mathrm{Size}(B) - \mathrm{Size}(A \cap B)$$

$$\mathrm{Size}(A \times B) = \mathrm{Size}(A)\ \mathrm{Size}(B)$$

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Examples of size-like notions

- Cardinality of set
- Dimension of vector space
- Measure of a measurable set
- Euler characteristic of topological space
- Intrinsic volumes of convex set (volume, surface area, total mean curvature, . . .)
- Capacity in potential theory / diversity of biological system
- Magnitude

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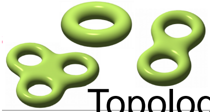
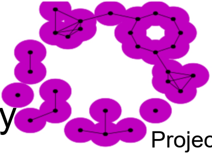
- Capacity in potential theory / diversity of biological system
- Magnitude (Leinster '08)



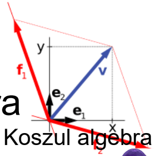
Topology

Triangulated manifolds:
Euler characteristic



Homology theory
≈ persistent homology

Algebra



Projective modules over Koszul algebra:
Euler form

metric spaces

number of points

Magnitude

Notion of size: How big is X?

Leinster, Doc. Math. (2008)
Euler characteristic for enriched categories

Ecology

Relates to diversity of ecosystem
(number of species)

Elliptic PDEs
Semiclassical Analysis
Geom. Measure Theory

→

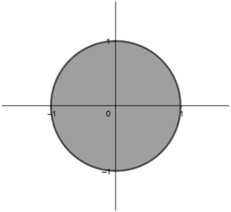
Geometry

Dim, volume, surface area, curvatures

Barceló, Carbery, Amer J. Math. (2017)
Gimperlein, Goffeng, Amer J. Math. (2021), + Louca (2021)
Leinster, Meckes, survey article (2017)

Outline of talk

What is the magnitude of the unit disk?



- What is magnitude?
- Magnitude of compact domains X in $\mathbb{R}^n$ and manifolds
- Leinster-Willerton conjecture: geometry of $\mathcal{M}_X(R) := \mathrm{mag}(R \cdot X)$
- Technically: semiclassical analysis of a pseudodifferential boundary problem for $(R^2 - \Delta)^{\pm(n+1)/2} + l.o.t.$

Magnitude of a matrix

Let $\mathcal{Z} \in \mathbb{R}^{n\times n}$. w **weighting** on $\mathcal{Z}$ if

$$\mathcal{Z}w = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} =: \mathbf{1}.$$

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Definition (Magnitude of a matrix)

If $\mathcal{Z}$ admits a weighting w and $\mathcal{Z}^T$ admits a weighting w',

$$\mathrm{mag}(\mathcal{Z}) := \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} w'_i.$$

Set $\mathrm{mag}(\mathcal{Z}) := +\infty$ otherwise.

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Check: $\mathrm{mag}(\mathcal{Z})$ is independent of choice of weighting $w, \tilde{w}$:

$$\mathrm{mag}(\mathcal{Z}) = \langle w, \mathbf{1}\rangle = \langle w, \mathcal{Z}^T w'\rangle = \langle \mathcal{Z}w, w'\rangle = \langle \mathcal{Z}\tilde{w}, w'\rangle = \langle \tilde{w}, \mathcal{Z}^T w'\rangle = \langle \tilde{w}, \mathbf{1}\rangle$$

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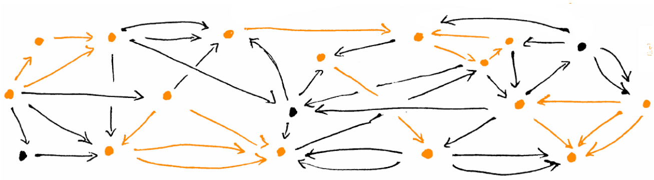
Check: $\operatorname{mag}(\mathcal{Z})$ is independent of choice of weighting.
If $\mathcal{Z}$ invertible, $\operatorname{mag}(\mathcal{Z}) = \langle w, \mathbf{1}\rangle = \langle \mathcal{Z}^{-1}\mathbf{1}, \mathbf{1}\rangle = \sum_{i,j}(\mathcal{Z}^{-1})_{ij}$.

Magnitude of (finite) categories and metric spaces

If $\mathcal{C}$ is a finite category, we set

$$\mathcal{Z}_{\mathcal{C}} := (\#\mathrm{Mor}_{\mathcal{C}}(i,j))_{i,j\in\mathrm{Obj}(\mathcal{C})} \quad \text{and} \quad \mathrm{mag}(\mathcal{C}) := \mathrm{mag}(\mathcal{Z}_{\mathcal{C}}).$$



(Figure by T.-D. Bradley, arxiv:1809.05923)

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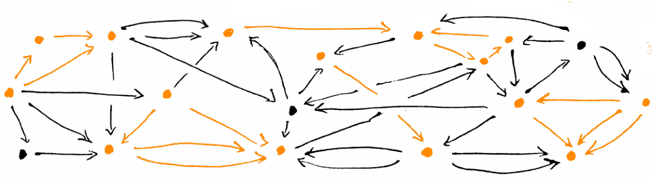
Theorem (Leinster, original motivation for magnitude)

Let $\mathcal{C}$ denote a finite category and $B\mathcal{C}$ its classifying space, then

$$\chi(B\mathcal{C}) = \mathrm{mag}(\mathcal{C}).$$

Similarly, mag recovers Euler characteristic of a triangulated manifold.



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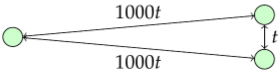
Examples

- One point space: $\mathrm{mag}(\cdot) = 1$
- Discrete "metric" space X: $\mathrm{mag}(X) = |X|$
- (Complete graph) N points, distance R:

$$\mathrm{mag}(\overbrace{\cdot \leftarrow R \rightarrow \cdot \quad \cdots \quad \cdot}^{N}) = \frac{N}{1+(N-1)\mathrm{e}^{-R}}$$

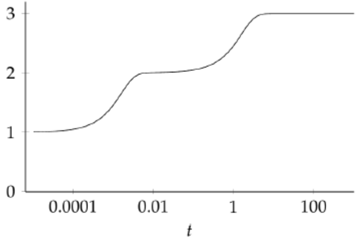
Example: a 3-point space (Simon Willerton)

Take the 3-point space

$X =$ 

- When t is small, X looks like a 1-point space.
- When t is moderate, X looks like a 2-point space.
- When t is large, X looks like a 3-point space.

Indeed, the magnitude of X as a function of t is:



(Slide by T. Leinster.)

The magnitude function of a metric space

If (X, d) is a finite metric space, we set

$$\mathcal{Z}_X := (\mathrm{e}^{-d(a,b)})_{a,b\in X} \quad \text{and} \quad \mathrm{mag}(X, d) := \mathrm{mag}(\mathcal{Z}_X).$$

The magnitude function of (X, d) is given by

$$\mathcal{M}_X(R) = \mathrm{mag}(X, R \cdot d) \qquad (R > 0)\ .$$

Observation

- $\mathcal{M}_X$ extends meromorphically to $R \in \mathbb{C}$
- $\mathcal{M}_X(R) = |X| + O(R^{-\infty})$ as $R \to \infty$
- $\mathcal{M}_X(0) = 1$

The magnitude function of a compact metric space

(X, d) compact metric space, $M(X)$ finite Borel measures on X.

$$\mathcal{Z}_X(R) : M(X) \to C(X), \quad \mathcal{Z}_X(R)\mu(x) := \int_X \mathrm{e}^{-R\mathrm{d}(x,y)}\mathrm{d}\mu(y).$$

A weighting measure μ_R is a solution to the equation $\mathcal{Z}_X(R)\mu_R = 1$ on X.

Definition / Theorem (Meckes)

Let (X, d) a positive definite compact metric space, i.e. $\mathcal{Z}_A(R)$ is a positive definite matrix for all finite $A \subseteq X$. Then

$$\mathcal{M}_X(R) := \sup\{\mathcal{M}_A(R) : A \subseteq X \text{ finite}\}$$

If (X, d) admits a weighting measure, $\mathcal{M}_X(R) = \mu_R(X)$.

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Explicitly for $X \subset \mathbb{R}^n$, $\mathcal{Z}_X(R)f(x) := \int_{\mathbb{R}^n} \mathrm{e}^{-R|x-y|} f(y)\mathrm{d}V(y) = g_R * f(x)$, where $g_R(x) := \mathrm{e}^{-R|x|}$. With $\hat{g}_R(\xi) = n!\omega_n R(R^2 + |\xi|^2)^{-(n+1)/2}$,

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The function $h := \mathcal{Z}_X(R)\mu_R$ extends to $\mathbb{R}^n$, with $h = 1$ on X.

Theorem (Meckes)

$$\mathcal{M}_X(R) = \frac{1}{n!\omega_n R} \inf\left\{ \|(R^2 - \Delta)^{(n+1)/4} h\|^2_{L^2(\mathbb{R}^n)} : h \in H^{(n+1)/2}(\mathbb{R}^n),\ h = 1 \text{ on } X \right\}$$
$$= \frac{1}{n!\omega_n R} \|(R^2 - \Delta)^{(n+1)/4} h\|^2_{L^2(\mathbb{R}^n)}$$

where $h \in H^{(n+1)/2}(\mathbb{R}^n)$ solves

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Example

$X = [-1,1] \subset \mathbb{R}$. $\mathcal{M}_X(R) = 1 + R = \chi(X) + \frac{\mathrm{Length}(X)}{2}R$.

Geometry of the magnitude function

Conjecture (Leinster, Willerton)

Let $X \subseteq \mathbb{R}^n$ be a compact convex subset. Then

$$\mathcal{M}_X(R) = \sum_{k=0}^{n} \frac{V_k(X)}{k!\omega_k} R^k \ .$$

Here $V_k(X)$ is the k-th intrinsic volume of X, ω_k volume of unit ball in $\mathbb{R}^k$.

For X a smooth convex body: $V_n(X) = \mathrm{vol}_n(X)$, $V_{n-1}(X) = \mathrm{vol}_{n-1}(\partial X)$, $V_{n-2}(X) = \int_{\partial X} H \mathrm{d}S$, ..., $V_0(X) = \chi(X)$.

Special cases:
$n = 1$: $\mathcal{M}_X(R) = \chi(X) + \frac{\mathrm{Length}(X)}{2} R$.
$n = 2$: $\mathcal{M}_X(R) = \chi(X) + \frac{\mathrm{Perim}(\partial X)}{4} R + \frac{\mathrm{Area}(X)}{2\pi} R^2$.

The conjecture was motivated by computational examples, the historical analogy with the Euler characteristic, and Hadwiger's theorem.

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Consequences include: Inclusion-Exclusion principle

$$\mathcal{M}_{A\cup B} = \mathcal{M}_A + \mathcal{M}_B - \mathcal{M}_{A\cap B} \ .$$

Conjecture true for $X \subset \mathbb{R}$

$X = [-1,1] \subset \mathbb{R}$. $\mathcal{M}_X(R) = 1 + R = \chi(X) + \frac{\mathrm{Length}(X)}{2} R$.

Leinster-Willerton conjecture: (Counter) Examples

Barcelo, Carbery 2017 prove the conjectured behaviour for $R \to \infty$ and $R \to 0$:

Theorem

Let $X \subseteq \mathbb{R}^n$ be a compact smooth domain. Then

$$\mathcal{M}_X(R) = \frac{\mathrm{vol}(X)}{n!\omega_n} R^n + o(R^n), \text{ as } R \to \infty,$$

$$\lim_{R\to 0} \mathcal{M}_X(R) = 1.$$

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Furthermore, they solve the magnitude PDE for $X = B(0,1) \subseteq \mathbb{R}^n$, $n = 5$:

$$\mathcal{M}_X(R) = \frac{1}{n!\omega_n R} \|(R^2 - \Delta)^{(n+1)/4} h\|^2_{L^2(\mathbb{R}^n)}$$

where $h \in H^{(n+1)/2}(\mathbb{R}^n)$ satisfies

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The result is

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Counterexample to Leinster-Willerton conjecture (Barcelo, Carbery 2017)

Let $X = B(0,1) \subseteq \mathbb{R}^5$ be the unit ball. Then

$$\mathcal{M}_X(R) = \frac{R^5}{5!} + \frac{3R^5 + 27R^4 + 105R^3 + 216R^2 + 72}{24(R+3)}$$

Not a polynomial. Also coefficients of R^k wrong.

Willerton obtains similar rational functions for balls in odd dimensions.

Asymptotic Leinster-Willerton conjecture

With M. Goffeng, we studied the magnitude PDE for general X, n odd:

$$(R^2 - \Delta)^{(n+1)/2} h = 0 \quad \text{weakly in } \mathbb{R}^n \setminus X, \quad h = 1 \text{ on } X.$$

Using Green's formula, for suitable boundary traces $\mathcal{D}_R^k h|_{\partial X}$ of order k:

$$\begin{aligned}
\mathcal{M}_X(R) &= \frac{1}{n!\omega_n R} \|(R^2 - \Delta)^{(n+1)/4} h\|^2_{L^2(\mathbb{R}^n)} \\
&= \frac{\mathrm{vol}_n(X)}{n!\omega_n} R^n - \frac{1}{n!\omega_n} \sum_{\frac{m}{2} < j \leq m} R^{n-2j} \int_{\partial X} \mathcal{D}_R^{2j-1} h \, dS \\
&= \frac{\mathrm{vol}_n(X)}{n!\omega_n} R^n - \frac{1}{n!\omega_n} \sum_{\frac{m}{2} < j \leq m} R^{n-2j} \int_{\partial X} \Lambda_{2j-1}(R) \, 1 \, dS.
\end{aligned}$$

The Λ_k are components of an $m \times m$ Dirichlet-to-Neumann operator, a parameter-elliptic pseudodifferential operator on ∂X, meromorphic in R. Analytic Fredholm theory and an explicit symbol computation show:

Asymptotic Leinster-Willerton conjecture

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Theorem (Gimperlein, Goffeng, Amer J Math 2021 $+\epsilon$)

Let $X \subseteq \mathbb{R}^n$ be compact with smooth boundary and $n = 2m - 1$.

- $\mathcal{M}_X$ extends meromorphically to $\mathbb{C}$.
- $\mathcal{M}_X$ holomorphic in sector $\{|\arg(z)| < \frac{\pi}{n+1}\}$.
  Finite number of poles in any sector $\{|\arg(z)| < \alpha\}$, $\alpha < \frac{\pi}{2}$.
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- There are constants $(c_k(X))_{k\in\mathbb{N}}$ such that for all $N \in \mathbb{N}$

  $$\mathcal{M}_X(R) = \frac{1}{n!\omega_n} \sum_{k=0}^{n+N} c_k(X) R^{n-k} + O(R^{-N}).$$

  The first four coefficients are given by

  $$c_0(X) = \mathrm{vol}_n(X),\ c_1(X) = m\,\mathrm{vol}_{n-1}(\partial X),$$

  $$c_2(X) = \frac{m^2}{2}\,(n-1)\int_{\partial X} H\,dS, \qquad (H \text{ mean curvature of } \partial X)$$

  $$c_3(X) = \alpha_n \int_{\partial X} H^2\,dS \qquad \text{(Willmore energy)}$$

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$$c_3(X) = \alpha_n \int_{\partial X} H^2\, dS \qquad (\text{Willmore energy, not intrinsic volume})$$

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$$c_2(X) = \frac{m^2}{2}(n-1)\int_{\partial X} H\, dS, \qquad c_3(X) = \alpha_n \int_{\partial X} H^2\, dS$$

- For $j \geq 4$, the coefficient $c_j(X)$ is an integral over ∂X of a universal polynomial in covariant derivatives of the fundamental form of total order $j-2$ and total degree $j-1$.

Corollaries

Asymptotic inclusion-exclusion

Let $n = 2m - 1$. If $A, B \subseteq \mathbb{R}^n$, as well as $A \cup B$ and $A \cap B$ are smooth compact domains then

$$\mathcal{M}_{A\cup B}(R) = \mathcal{M}_A(R) + \mathcal{M}_B(R) - \mathcal{M}_{A\cap B}(R) + O(R^{-\infty}).$$

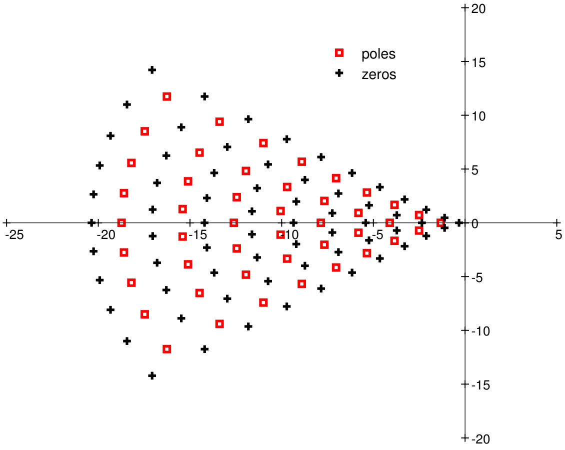
Definite failure of original Leinster-Willerton conjecture

The coefficient $c_3(X) = \alpha_n \int_{\partial X} H^2 \mathrm{d}S$ is not Hausdorff continuous and not an intrinsic volume. Indeed, for $n = 3$ by homogeneity $c_3(X)$ would need to be proportional to the Euler characteristic. But $c_3 =$ Willmore energy can be made arbitrarily large on surfaces of genus zero.

Can you "magnitude" the shape of a drum?

Let X as in theorem, B ball. If $\mathcal{M}_X \sim \mathcal{M}_B$, then X is isometric to B (use asymptotics & isoperimetric inequality).
There are nonconvex domains $X, Y =$ balls with a hole which are not isometric, but $\mathcal{M}_X = \mathcal{M}_Y$ (Meckes).

Poles and zeros of the magnitude function $\mathcal{M}_{B_{21}}$
B_{21} = unit ball in dimension $n = 21$



Poles and zeros of $\mathcal{M}_{B_n}$ in dimensions $n = 13, 17, 21$

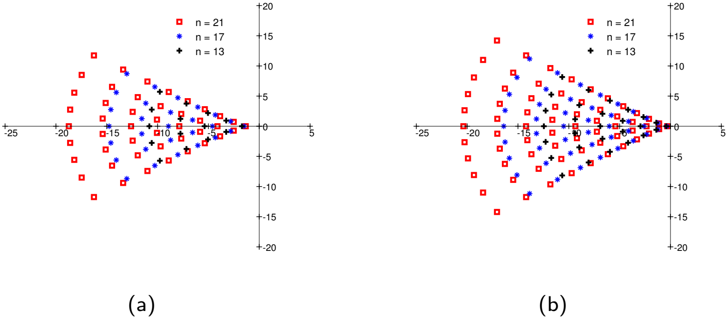
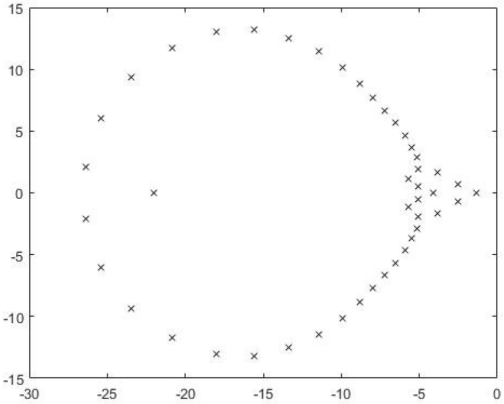
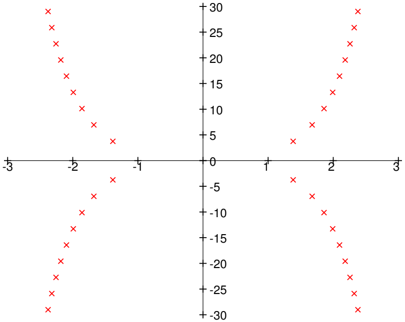


Figure: (a) poles, (b) zeros.

Computer algebra delicate: Naive computation of poles of $\mathcal{M}_{B_{21}}$ (with Maple, Sage)



Poles of $\mathcal{M}_X$ for spherical shell in dimension 3 $X = (2B_3) \setminus B_3^\circ$



infinite number of poles approaching curve $|\mathrm{Re}(R)| = \log(|\mathrm{Im}(R)|)$

$$\mathcal{M}_X(R) = \frac{7}{6}R^3 + 5R^2 + 2R + 2 + \frac{\mathrm{e}^{-2R}(R^2+1) + 2R^3 - 3R^2 + 2R - 1)}{\sinh(2R) - 2R}$$

Asymptotic Leinster-Willerton conjecture

Theorem (Gimperlein, Goffeng, Amer J Math 2021 $+\epsilon$)

Let $X \subseteq \mathbb{R}^n$ be compact with smooth boundary and $n = 2m - 1$.

- $\mathcal{M}_X$ extends meromorphically to $\mathbb{C}$.
- $\mathcal{M}_X$ holomorphic in sector $\{|\arg(z)| < \frac{\pi}{n+1}\}$.
  Finite number of poles in any sector $\{|\arg(z)| < \alpha\}$, $\alpha < \frac{\pi}{2}$.
- There are constants $(c_k(X))_{k\in\mathbb{N}}$ such that for all $N \in \mathbb{N}$

$$\mathcal{M}_X(R) = \frac{1}{n!\omega_n} \sum_{k=0}^{n+N} c_k(X) R^{n-k} + O(R^{-N}).$$

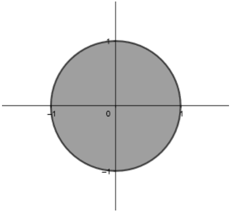
  The first four coefficients are given by

$$c_0(X) = \mathrm{vol}_n(X),\ c_1(X) = m\,\mathrm{vol}_{n-1}(\partial X),$$
$$c_2(X) = \frac{m^2}{2}(n-1)\int_{\partial X} H\, dS, \qquad c_3(X) = \alpha_n \int_{\partial X} H^2\, dS$$

- For $j \geq 4$, the coefficient $c_j(X)$ is an integral over ∂X of a universal polynomial in covariant derivatives of the fundamental form of total order $j-2$ and total degree $j-1$.

What is the magnitude of the unit disk?

In even dimensions almost nothing has been known about magnitude.



$$\mathcal{M}_{B_2(0,1)}(R) = ?$$

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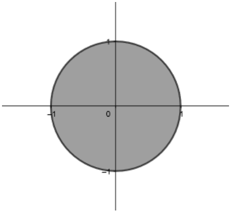

Theorem (Gimperlein, Goffeng, Louca 2021)

$$\mathcal{M}_{B_2(0,1)}(R) = \frac{1}{2}R^2 + \frac{3}{2}R + O(1)$$

More generally, we extend the previous statements on the meromorphic continuation and asymptotic expansion of $\mathcal{M}_X$ to smooth compact domains $X \subset M$, where $M = \mathbb{R}^n$ or (under technical assumptions) M manifold with metric.

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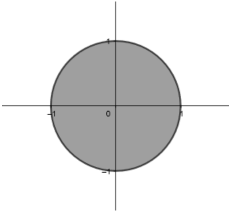

Theorem (Gimperlein, Goffeng, Louca 2021)

$$\mathcal{M}_{B_2(0,1)}(R) = \frac{1}{2}R^2 + \frac{3}{2}R + O(1)$$

- Study directly the boundary problem for $\mathcal{Z}_X(R) = (R^2 - \Delta)^{-(n+1)/2}$:

$$\mathcal{Z}_X(R)\mu_R = 1 \quad \text{in } X, \quad \mathrm{supp}\mu_R \subseteq X.$$

- Analysis relies on ideas of Hörmander, Eskin, as recently developed by Grubb for fractional boundary problems involving $(-\Delta)^s$, $s > 0$.

Set-up

- $X \subseteq \mathbb{R}^n$ compact domain with smooth boundary (more generally: $X \subseteq \mathbb{R}^N$ compact submanifold with boundary).
- Consider the operator

$$\tilde{\mathcal{Z}}_X(R)f(x) = \frac{1}{R}\mathcal{Z}_X(R)f(x) = \frac{1}{R}\int_X \mathrm{e}^{-Rd(x,y)} f(y)\, dV(y)$$

- For $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^n) := \{u : (1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\};$$
$$\mathring{H}^s(X) := \{u \in H^s(\mathbb{R}^n) : \mathrm{supp}(u) \subseteq X\};$$
$$\bar{H}^s(X) := H^s(\mathbb{R}^n)/\mathring{H}^s(\mathbb{R}^n \setminus X) = \{u|_X : u \in H^s(\mathbb{R}^n)\}.$$

- $\mathring{H}^0(X) = \bar{H}^0(X) = L^2(X)$ and the L^2-pairing extends to a perfect pairing

$$\mathring{H}^s(X) \times \bar{H}^{-s}(X) \to \mathbb{C}.$$

Some facts

- $\tilde{\mathcal{Z}}_X(R)$ is a *parameter-elliptic pseudodifferential operator* of order $-n-1$.
- $\tilde{\mathcal{Z}}_X(R) : \mathring{H}^{-(n+1)/2}(X) \to \bar{H}^{(n+1)/2}(X)$ is a continuous isomorphism for $\mathrm{Re}(R) \gg 0$, which extends to a holomorphic Fredholm operator valued function of $R \in \mathbb{C}$.
- Fredholm theory: $\tilde{\mathcal{Z}}_X(R)^{-1} : \bar{H}^{(n+1)/2}(X) \to \mathring{H}^{-(n+1)/2}(X)$ extends meromorpically to $R \in \mathbb{C}$. It is "computable" up to $O(R^{-\infty})$.
- The function $h \in H^{(n+1)/2}(\mathbb{R}^n)$ defined from

$$h(x) := R^{-1} \int_X \mathrm{e}^{-R|x-y|}(\tilde{\mathcal{Z}}_X(R)^{-1} 1_X)(y) \mathrm{d}V(y),$$

solves Meckes' minimization problem, so that

$$\mathcal{M}_X(R) = R^{-1}(\tilde{\mathcal{Z}}_X(R)^{-1} 1_X, 1_X)_{L^2}.$$

The magnitude function thus extends meromorphically to $R \in \mathbb{C}$.

Construction of asymptotic expansions

Theorem (Gimperlein-Goffeng-Louca '21)

Let $X \subseteq \mathbb{R}^n$ be a compact domain with smooth boundary. Then

$$\mathcal{M}_X(R) = \frac{1}{n!\omega_n} \sum_{k=0}^{\infty} c_k(X) R^{n-k} + O(R^{-\infty}).$$

The first three coefficients are given by

$$c_0(X) = \mathrm{vol}_n(X),\ c_1(X) = \gamma_{n,1}\,\mathrm{vol}_{n-1}(\partial X),\ c_2(X) = \gamma_{n,2} \int_{\partial X} H \mathrm{d}S,$$

where H is the mean curvature. There exists an iterative way of computing the coefficients $c_k(X)$.

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- Iterative scheme is a result of a Wiener-Hopf factorization using a 'nice' factorization of the symbol.
- The result holds for $X \subseteq \mathbb{R}^N$ a compact submanifold with boundary, and in this case

$$c_k(X) = \int_X \alpha_k(x) \mathrm{d}V(x) + \int_{\partial X} \beta_k(x) \mathrm{d}S(x),$$

and $\alpha_k = 0$ for all odd k.
There is an iterative scheme to compute α_k and β_k.

The idea for $X \subseteq \mathbb{R}$

Consider $X = [0,1] \subseteq \mathbb{R}$ so we are looking for a $u \in \mathring{H}^{-1}(0,1)$ with

$$\mathcal{Z}_X(R)u(x) = \int_0^1 \mathrm{e}^{-R|x-y|}u(y)\mathrm{d}y = 1.$$

Fourier transforming gives

$$\mathcal{F}(\mathcal{Z}_X(R)u)(\xi) = 2R(R^2+\xi^2)^{-1}\hat{u}(\xi) = 2R(R+i\xi)^{-1}(R-i\xi)^{-1}\hat{u}(\xi).$$

Elementary computations give

$$\begin{cases} (R+i\xi)^{-1} = \mathcal{F}(z_R^+), & z_R^+(x) = \chi_{[0,\infty)}(x)\mathrm{e}^{-Rx} \\ (R-i\xi)^{-1} = \mathcal{F}(z_R^-), & z_R^-(x) = \chi_{(-\infty,0]}(x)\mathrm{e}^{Rx} \end{cases}$$

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Paley-Wiener!

Consider $\mathcal{Z}_X(R)^{\pm} f := z_R^{\pm} * f$.

The idea for $X \subseteq \mathbb{R}$, continued

The operators $\mathcal{Z}_X(R)^{\pm} f := z_R^{\pm} * f$ satisfy for any $a \in \mathbb{R}$:

$$\begin{cases} \mathrm{supp}(f) \subseteq [a, \infty) \Rightarrow \mathrm{supp}(\mathcal{Z}_X(R)^+ f) \subseteq [a, \infty), \\ \mathrm{supp}(f) \subseteq (-\infty, a] \Rightarrow \mathrm{supp}(\mathcal{Z}_X(R)^- f) \subseteq (-\infty, a]. \end{cases}$$

Therefore, for any real s and a, $\mathcal{Z}_X(R)^{\pm}$ defines isomorphisms

$$\begin{cases} \mathcal{Z}_X(R)^+ : \mathring{H}^s(a, \infty) \to \mathring{H}^{s+1}(a, \infty), \; \mathcal{Z}_X(R)^+ : \overline{H}^s(-\infty, a) \to \overline{H}^{s+1}(-\infty, a), \\ \mathcal{Z}_X(R)^- : \mathring{H}^s(-\infty, a) \to \mathring{H}^{s+1}(-\infty, a), \; \mathcal{Z}_X(R)^- : \overline{H}^s(a, \infty) \to \overline{H}^{s+1}(a, \infty). \end{cases}$$

Problem arising in the general setting

How to factor $\mathcal{Z}_X(R) : \mathring{H}^{-1}(0,1) \to \overline{H}^1(0,1)$ *near* ∂X as mapping

$$\mathring{H}^{-1}(0,1) \xrightarrow{\mathcal{Z}_{X,+}(R)} \mathring{H}^0(0,1) = L^2(0,1) = \overline{H}^0(0,1) \xrightarrow{\mathcal{Z}_{X,-}(R)} \overline{H}^1(0,1)?$$

Structure of inverse operator

$$\mathcal{Z}_X(R)^{-1} = \tilde{\chi}_1 Q^{-1} \chi_1 + \tilde{\chi}_2^{-1} \mathcal{Z}_{X,+}(R)^{-1} \mathcal{Z}_{X,-}(R)^{-1} \chi_2 + S$$

where χ_j gluing functions, Q^{-1} *interior parametrix*, $S = O(R^{-\infty})$.

The idea for $X \subseteq \mathbb{R}$, continued

Formally, $(\mathcal{Z}_X(R)^{\pm})^{-1} = R \pm \partial_x$. Is it simply that

$$\mathcal{Z}_X(R)^{-1} f = \frac{1}{2R}(R + \partial_x)(R - \partial_x) f?$$

For $f = 1_{[0,1]} \in \overline{H}^1(0,1)$, then

$$\frac{1}{2R}(R + \partial_x)(R - \partial_x)1_{[0,1]} = \frac{1}{2R}(R + \partial_x)(\underbrace{R1_{[0,1]}}_{\in \mathring{H}^0(0,1)}) = \frac{R}{2}1_{[0,1]} + \frac{1}{2}(\delta_{x=1} - \delta_{x=0}).$$

But we have that

$$\mathcal{Z}_X(R)\left(\frac{R}{2}1_{[0,1]} + \frac{1}{2}(\delta_{x=1} - \delta_{x=0})\right) = 1 - \mathrm{e}^{-R(1-x)},$$

It does not hold that $\mathrm{e}^{-R(1-x)} = O(R^{-\infty})$ in norm sense on $\overline{H}^1(0,1)$!
The correct answer is

$$\mathcal{Z}_X(R)^{-1}1_{[0,1]} = \frac{R}{2}1_{[0,1]} + \frac{1}{2}(\delta_{x=1} + \delta_{x=0}).$$

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To fix the sign mistake: build $\mathcal{Z}_{X,\pm}(R)^{-1}$ from gluing $(\mathcal{Z}_X(R)^{\pm})^{-1}$ at $x = 0$ with $(\mathcal{Z}_X(R)^{\mp})^{-1}$ at $x = 1$... Morally we have that

$\mathcal{Z}_X(R)^{-1} = \frac{1}{2R}(R+\partial_x)(R-\partial_x)$ at $x = 1$, $\mathcal{Z}_X(R)^{-1} = \frac{1}{2R}(R-\partial_x)(R+\partial_x)$ at

Extension to manifolds

Compact, smooth $X \subseteq M$ for d satisfying a technical assumption, e.g.

- M is a sphere with geodesic distance
- M is a Riemannian manifold with geodesic distance and $\mathrm{diam}(X) < \mathrm{inj}(M)$
- M is a submanifold of $\mathbb{R}^n$ with the subspace metric

For large R, $\tilde{\mathcal{Z}}_R = n!\omega_n(R^2 - \Delta)^{-\frac{n+1}{2}} + \mathrm{l.o.t}(R, \mathrm{derivatives})$

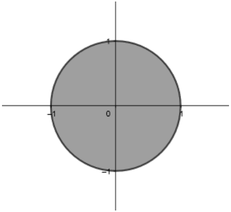
Some open analytic problems for magnitude

- For general Riemannian manifolds an improved understanding of the cut-locus seems required.
- The magnitude function for domains with edges: $\mathcal{M}_{X\cap Y}$?
- Poles of $\mathcal{M}_X$ analogous to scattering resonances.
  Interpretation: Why is there a pole at $R = -3$ for $B(0,1) \subset \mathbb{R}^5$?
  Counting: Sharp upper and lower bounds? Does a generic perturbation of $B(0,1) \subset \mathbb{R}^{2m-1}$ have infinitely many poles?
- Geometric interpretation of the Taylor coefficients for $\mathcal{M}_X$ at $R = 0$?
  Meckes (2020) proves *upper bounds* in terms of intrinsic volumes.

Magnitude is just one example of a semiclassical pseudodifferential boundary problem. Related questions arise for log-gases, random matrices, optimal placement problems, . . .

At beginning of talk I asked:

What is the magnitude of the *unit* disk?



This talk: qualitative properties and semiclassical limit for large radius.

Can one find an exact formula for the solution of the boundary problem for $(1-\Delta)^{3/2}$ outside the disk?

Thank you for your attention!

More details in: arXiv:1706.06839
and more to come very soon

Magnitude bibliography: `maths.ed.ac.uk/~tl/magbib`