Solutions to semilinear wave equations of very low regularity

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Abstract

This paper observes new phenomena for the wellposedness and propagation of singularities for semilinear wave equations with p-th power nonlinearity for initial data of very low Sobolev-regularity. In one space dimension we obtain solutions whose singular support propagates along any ray outside the light cone. These solutions exist for any Sobolev exponent $s < \frac{1}{2}$ in space, while the singular support of any solution of higher regularity is contained in the light cone. Motivated by these examples, we study wellposedness of semilinear wave equations for Sobolev data whose Fourier transform is supported in a half-line. Our result improves the wellposedness results for Sobolev data without the support condition and, in some cases, obtains wellposedness below $L^2(\mathbb{R})$. Extensions to higher space dimensions are given.

1 Introduction

This paper observes new phenomena for the wellposedness and propagation of singularities for semilinear wave equations with initial data of very low Sobolev-regularity. We address the problem

$$\partial_t^2 u - \Delta u = \pm u^p, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), \tag{1}$$

in space dimension n, where $p \geq 2$ is assumed to be a positive integer. The specialization to this case is needed for two reasons. First, we wish to study propagation of singularities from the initial data, and hence we need a smooth nonlinearity in order to avoid the occurrence of additional singularities. Second, we will use the Hörmander product of distributions, so only integer powers are amenable.

It is a general principle in linear wave propagation that sharp wave crests (singularities) propagate along light cones, or more precisely, along the bicharacteristics of the linear wave operator $\partial_t^2 u - \Delta u$. For semilinear wave equations, singularities of sufficiently smooth solutions are known to propagate along unions of bicharacteristics, where in space dimension $n > 1$ new singularities may arise at points of intersection of incoming wave crests. In dimension $n = 1$, a fundamental result for problem (1) assures that the singularities of any solution in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ propagate only along light cones [35, 38].

In this article we obtain solutions in $\mathcal{C}([-T,T] : H^s_{\mathrm{loc}}(\mathbb{R}))$, for any $s < \frac{1}{2}$, whose singular support lies outside the light cone. As $\mathcal{C}([-T,T] : H^s_{\mathrm{loc}}(\mathbb{R})) \subset L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ when $s > \frac{1}{2}$, they establish a sharp threshold $s = \frac{1}{2}$ for the Sobolev exponent between solutions with expected, respectively anomalous, propagation of singularities. Note that the maximal regularity of solutions with unexpected properties has attracted significant recent interest for equations from continuum mechanics and geometry [11].

For the anomalous solutions u presented here, at fixed time t the Fourier transform $\widehat{u}$ with respect to x is supported in a half-line. Motivated by this fact, we also extend the range of wellposedness for

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problem (1) to data and solutions u with this property. The main result of this article is

Theorem. Let $n = 1$ and $H^s_\Gamma(\mathbb{R}) = \{f \in H^s(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [0, \infty)\}$.

(a) (Anomalous propagation) For any $c \neq \pm 1$ there exist solutions to (1) with singular support along the line $\{x + ct = 0, t \in \mathbb{R}\}$, i.e., along any ray off the light cone. More prescisely, for any $c \neq \pm 1$ and any $s < \frac{1}{2}$ there are H^s_{loc}-solutions with this property.

(b) (Low regularity wellposedness) Problem (1) is wellposed in $H^s_\Gamma(\mathbb{R})$ for $p = 2$ and $s > -\frac{1}{2}$ and for $p \geq 3$ and $s > \frac{1}{2} - \frac{1}{2p-4}$.

Part (a) combines Proposition 3 with the discussion in Section 4. The construction builds on recent examples found by one of the authors [24]; for any $s < \frac{1}{2}$ there is such a solution in $\mathcal{C}([-T, T] : H^s_{\mathrm{loc}}(\mathbb{R})) \cap \mathcal{C}^1([-T, T] : H^{s-1}_{\mathrm{loc}}(\mathbb{R}))$, if p is large enough. Part (b) is the content of Theorem 9. Note that it improves the wellposedness results for data in $H^s(\mathbb{R})$ without the support condition, where problem (1) is wellposed if $s > \frac{1}{2} - \frac{1}{p}$ and illposed if $s < \frac{1}{2} - \frac{1}{p}$. The solutions constructed in (a) do not fall into the known H^s-wellposedness regimes.

Section 7 addresses the extension of this Theorem to higher space dimensions $n > 1$. We give examples of anomalous solutions to (1) which belong to $\mathcal{C}([-T, T] : H^s_{\mathrm{loc}}(\mathbb{R}^n)) \cap \mathcal{C}^1([-T, T] : H^{s-1}_{\mathrm{loc}}(\mathbb{R}^n))$ for $s < \frac{n}{2}$. A similar extension of the wellposedness theory remains open.

The remainder of this introduction is devoted to a literature review of propagation of singularities and of critical Sobolev exponents.

The investigation of propagation of singularities in semilinear hyperbolic equations and systems started with the discovery of Jeffrey Rauch and Michael Reed [35, 37] that – unlike in the linear case – singularities may arise that cannot be traced back via bicharacteristics to singularities in the initial data, but may be produced at later times by the interaction of singularity bearing bicharacteristics. For a survey of the huge number of results up to around 1990 we refer to the monograph [4]. Rauch and Reed coined the term *anomalous singularities* for this phenomenon. However, these "anomalous singularities" still propagated along characteristics/bicharacteristics, as opposed to the noncharacteristic singularities in the present paper, which are even more anomalous.

There is one exception, namely the wave equation $\partial_t^2 u - \partial_x^2 u = f(u)$ with smooth nonlinearity $f(\cdot)$ in one space dimension (actually any (2×2)-first order system in $n = 1$) where the propagation is as in the linear case. This is due to the fact that there are only two characteristic directions, thereby avoiding nonlinear interaction at later times. Here the results of [35, 38] say that for distributional solutions to the semilinear wave equation which belong to $L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ no anomalous singularities arise. (This applies, in particular, to solutions which belong to $\mathcal{C}([-T, T] : H^s(\mathbb{R}))$ with $s > \frac{1}{2}$.) For example, if the singular support of the initial data is $\{x = 0\}$ then the solution is smooth except possibly along the light cone $\{|x| = |t|\}$.

In higher space dimensions, the first and prototypical result is due to Rauch [33]. It says the following: Suppose that u is a distributional solution to (1) (even with a polynomial nonlinearity) which belongs to $H^s_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ with $s > (n + 1)/2$ and let the initial data belong to $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$. Then u is $\mathcal{C}^\infty$ on $\{|x| > |t|\}$, and it belongs to $H^{s+1+\sigma}_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ on $\{|x| < |t|\}$ for all $\sigma < s - (n + 1)/2$. It is also known that the singular support of the solution may contain the solid cone $\{|x| \leq |t|\}$, see [1], where an example is given with Sobolev regularity just above $3s - n + 2$ in $\{|x| < |t|\}$.

These results date back to a time when the investigation of critical exponents had not yet been picked up. Accordingly, the usual setting was in $H^s_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ with $s > (n + 1)/2$, in which case H^s_{loc} is an algebra. The methods were commonly based on a microlocal analysis of the nonlinear action [5, 36], as well as on paradifferential calculus [7]. Few papers addressed propagation of local regularity in lower Sobolev regularity, as the paper [14] which went as low as $s > 0$ (but still requiring L^∞_{loc}); see also the early counterexamples of anomalous bicharacteristic behavior in low regularity in the second part of [34].

In the meantime the wellposedness of problem (1) for data of low regularity has been clarified. Recall that problem (1) is locally wellposed in H^s if, for every $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$, there is $T > 0$ and a unique distributional solution u belonging to $\mathcal{C}([-T,T] : H^s(\mathbb{R}^n)) \cap \mathcal{C}^1([-T,T] : H^{s-1}(\mathbb{R}^n))$. Further, u is required to belong to a space on which the p-th power is welldefined (usually $L^p_{\mathrm{loc}}(\mathbb{R}^{n+1})$), and the map $(u_0, u_1) \to u$ should be continuous. The p-th power here may also be understood as a Fourier product, see Section 3.

As summarized in [9, 12], the critical regularity for local H^s-wellposedness of problem (1) is

$$s_{\mathrm{crit}} = \max\left(\frac{n}{2} - \frac{2}{p-1}, \frac{n+1}{4} - \frac{1}{p-1}, 0\right).$$

Essentially, wellposedness has been established for $s \geq s_{\mathrm{crit}}$, possibly with additional constraints in certain ranges of p and n, while illposedness has been proven for $s < s_{\mathrm{crit}}$, again with certain gaps in the ranges. Relevant literature is [17, 19, 20, 21, 22, 41], as well as recent directions for the probabilistic wellposedness [8, 25, 26, 27, 43]. For more details, the reader is referred to the summaries in [9, 12]. The case $n = 1$ deserves special attention. The critical exponent is

$$s_{\mathrm{sob}} = \max\left(\frac{1}{2} - \frac{1}{p}, 0\right). \tag{2}$$

The stronger bound is needed in order to have $H^s(\mathbb{R}) \subset L^p(\mathbb{R})$. It was shown in [9] that problem (1) is H^s-illposed for $s < s_{\mathrm{sob}}$. In addition, it was shown there that norm inflation takes place for $\frac{1}{2} - \frac{1}{p-1} < s < s_{\mathrm{sob}}$ and for $s \leq -\frac{1}{2}$. Further, the authors also showed that the solution map is discontinuous at $(0,0)$ for $s \leq \frac{1}{2} - \frac{1}{p-1}$. These results were complemented by [12] which proved norm inflation also in the range $s < 0$. It is also noted in [9] that problem (1) is locally H^s-wellposed when $n = 1$ and $s \geq s_{\mathrm{sob}}$.

2 Notation

The notation generally follows [40]. In particular, the Fourier transform is used in the form

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int \mathrm{e}^{-2\pi \mathrm{i} x\xi} f(x)\,\mathrm{d}x.$$

As usual, $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. For $s \in \mathbb{R}$, we write

$$L^2_s(\mathbb{R}^n) = \{h \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s h(\xi) \in L^2(\mathbb{R}^n)\}.$$

The Sobolev spaces and local Sobolev spaces, respectively, are defined by

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \widehat{f} \in L^2_s(\mathbb{R}^n)\}$$

and

$$H^s_{\mathrm{loc}}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \psi f \in H^s(\mathbb{R}^n) \text{ for all } \psi \in \mathcal{D}(\mathbb{R}^n)\}.$$

The distribution $(x+\mathrm{i}0)^\lambda \in \mathcal{S}'(\mathbb{R})$ is defined as

$$\lim_{\varepsilon\to 0}(x^2+\varepsilon^2)^{\lambda/2}\mathrm{e}^{\mathrm{i}\,\lambda\arg(x+\mathrm{i}\varepsilon)}.$$

It is an entire function of $\lambda \in \mathbb{C}$, see e.g. [13, Section I.3.6]. Its Fourier transform is given by

$$\mathcal{F}\big((x+\mathrm{i}0)^\lambda\big)(\xi) = \frac{(2\pi)^{-\lambda}}{\Gamma(-\lambda)}\mathrm{e}^{\mathrm{i}\,\lambda\pi/2}\xi_+^{-\lambda-1}$$

for $\lambda \neq 0, 1, 2, 3, \dots$ [13, Section II.2.3], where ξ_+^μ is the pseudofunction as defined in [13, Section I.3.2].

Remark 1. *The following properties are easy to show. We assume here that $\lambda < 0$ so that the pseudofunction $\xi_+^{-\lambda-1}$ is locally integrable. Then, for $\lambda < 0$, the following equivalences hold:*

(a) $(x+\mathrm{i}0)^\lambda \in H^s_{\mathrm{loc}}(\mathbb{R}) \;\Leftrightarrow\; s < \lambda + \frac{1}{2}$,

(b) $(x+\mathrm{i}0)^\lambda \in H^s(\mathbb{R}) \;\Leftrightarrow\; \lambda < -\frac{1}{2}$ and $s < \lambda + \frac{1}{2}$.

3 Multiplication of distributions

Let $S, T \in \mathcal{S}'(\mathbb{R}^n)$. The $\mathcal{S}'$-convolution of S and T is said to exist, if

$$(\varphi * S^-)T \in \mathcal{D}'_{L^1}(\mathbb{R}^n), \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where $S^-(x) = S(-x)$. In this case, the convolution is defined by $\langle S * T, \varphi\rangle = \langle(\varphi * S^-)T, 1\rangle$, and $S * T$ belongs to $\mathcal{S}'(\mathbb{R}^n)$.

Let $u, v \in \mathcal{S}'(\mathbb{R}^n)$. If the $\mathcal{S}'$-convolution of $\mathcal{F}u$ and $\mathcal{F}v$ exists, one may define the *Fourier product*

$$u \cdot v = \mathcal{F}^{-1}(\mathcal{F}u * \mathcal{F}v). \tag{3}$$

The definition can be localized as follows. Assume that for every $x \in \mathbb{R}^n$ there is a neighborhood Ω_x and $\chi_x \in \mathcal{D}(\mathbb{R}^n)$, $\chi_x \equiv 1$ on Ω_x, such that the $\mathcal{S}'$-convolution of $\mathcal{F}(\chi_x u)$ and $\mathcal{F}(\chi_x v)$ exists. Locally near x, the product $u \cdot v$ is defined to be $\mathcal{F}^{-1}(\mathcal{F}(\chi_x u) * \mathcal{F}(\chi_x v))$. Globally, it is defined by a partition of unity argument.

Remark 2. *Here are some special cases in which the Fourier product exists.*

(a) The existence of the $\mathcal{S}'$-convolution of $S, T \in \mathcal{S}'(\mathbb{R}^n)$ is guaranteed if both S and T have their support in a closed, acute and convex cone Γ in $\mathbb{R}^n$. Further, $S * T$ is also supported in Γ, and the map $(S, T) \to S * T$ is separately continuous in $\mathcal{S}'(\mathbb{R}^n)$ [44, I.5.6, I.4.5]. Let

$$\mathcal{S}'_\Gamma(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp}\widehat{f} \subset \Gamma\}.$$

For $u, v \in \mathcal{S}'_\Gamma(\mathbb{R}^n)$, the product $u \cdot v$ is thus definable by (3) and belongs to $\mathcal{S}'_\Gamma(\mathbb{R}^n)$. Thus $\mathcal{S}'_\Gamma(\mathbb{R}^n)$ forms an algebra with respect to multiplication, and the multiplication map is separately continuous.

(b) Let $u, v \in L^2(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $(\varphi * \widehat{u}^-)\widehat{v} \in L^1(\mathbb{R}^n) \subset \mathcal{D}'_{L^1}(\mathbb{R}^n)$. A simple calculation shows that the $\mathcal{S}'$-convolution $\widehat{u} * \widehat{v}$ exists and coincides with the ordinary convolution. By the exchange formula for L^2-functions, the Fourier product $u \cdot v$ coincides with the ordinary product of two L^2-functions. Using localization as indicated above, the same holds for the product of two L^2_{loc}-functions. In particular, for $u, v \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ with $s > n/2$, the Fourier product exists and coincides with the product in the algebra $H^s_{\mathrm{loc}}(\mathbb{R}^n)$.

When $1 \leq p < 2$, $2 < q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, there are examples of $u \in L^p(\mathbb{R})$, $v \in L^q(\mathbb{R})$ whose Fourier product does not exist, as shown in the recent paper [29]. However, if both the ordinary product and the Fourier product exist, they necessarily coincide.

(c) The product defined by Hörmander's wave front set criterion [15], requiring that for every $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, $(x, \xi) \in \mathrm{WF}(u)$ implies $(x, -\xi) \notin \mathrm{WF}(v)$, is also a special case. This can be seen by localizing the arguments establishing case (a), see e.g. [23, Proposition 6.3].

The remark shows that all products of functions and distributions occurring in this paper can be subsumed under the framework of the Fourier product. Further details and a discussion of different products of distributions can be found in [23].

4 Anomalous propagation of singularities to 1D-semilinear wave equations

In this section, we consider the propagation of singularities for the semilinear wave equation (1) in one dimension. The following proposition exhibits solutions with stationary singular support. They are used below to construct solutions whose singular support propagates in arbitrary noncharacteristic directions.

Proposition 3. *For every $s < \frac{1}{2}$ there are $\lambda < 0$ and $p \in \mathbb{N}$ such that*

(a) the distribution $u_0(x) = (x + \mathrm{i}0)^\lambda$ belongs to $H^s_{\mathrm{loc}}(\mathbb{R})$, $\operatorname{singsupp} u_0 = \{0\}$, and

(b) $u(x,t) \equiv u_0(x)$ is a distributional solution to the semilinear wave equation

$$\partial_t^2 u - \partial_x^2 u = -\lambda(\lambda - 1) u^p, \quad u(x, 0) = u_0(x),\ \partial_t u(x, 0) = 0 \tag{4}$$

where the nonlinear term is understood in the sense of the Fourier product. Its singular support is the noncharacteristic line $\{(0,t) : t \in \mathbb{R}\}$.

Similarly, for any $1 \neq p \in \mathbb{N}$ there are λ and $s < \frac{1}{2}$ such that $u(x,t) \equiv u_0(x)$ is a distributional solution of (1).

Multiplying the solution u of (4) by a constant, we obtain a corresponding solution of (1).

The special solutions exhibited here are self-similar solutions to the semilinear wave equation. However, they do not belong to the classes of functions considered e.g. in [6, 18, 30, 31, 39].

Proof of Proposition 3. The function $\mathbb{C} \to \mathcal{S}'(\mathbb{R})$, $\lambda \to (x + \mathrm{i}0)^\lambda$ is analytic and it is well-known that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x + \mathrm{i}0)^\lambda = \lambda(\lambda - 1)(x + \mathrm{i}0)^{\lambda - 2}$. The support of its Fourier transform is $[0, \infty)$, so all integer powers make sense by means of the Fourier product. Further,

$$(x + \mathrm{i}0)^{\lambda - 2} = (x + \mathrm{i}0)^{\lambda p}$$

provided $\lambda = \frac{2}{1-p}$. Noting that $\lambda < 0$, Remark 1 shows that $(x + \mathrm{i}0)^\lambda$ belongs to $H^s_{\mathrm{loc}}(\mathbb{R})$ iff $s < \lambda + \frac{1}{2} = \frac{2}{1-p} + \frac{1}{2}$. Let $p \to \infty$ to produce the desired s.

Finally, if $1 \neq p \in \mathbb{N}$, setting $\lambda = \frac{2}{1-p}$ produces a solution in $H^s_{\mathrm{loc}}(\mathbb{R})$ for $s < \frac{1}{2} - \frac{2}{p-1}$.

□

Remark 4. *(a) Anomalous propagation of singularities. When* $s > \frac{1}{2}$*, equation* (4) *with initial data in* $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *would have a unique solution which belongs to* $\mathcal{C}^0([-T,T] : H^s(\mathbb{R}^n)) \subset L^\infty(\mathbb{R} \times [-T,T])$*, so the anomalous singular support of the solution from Propositon 3 would be ruled out by the results in [35, 36].*

(b) Critical exponents. By Remark 1, $(x+\mathrm{i}0)^\lambda$ *with* $\lambda = \frac{2}{1-p}$ *actually belongs to* $H^s(\mathbb{R})$ *for* $p=2$ *and* $p=3$*, where* $s < \frac{2}{1-p} + \frac{1}{2}$*. However, this is outside the range of wellposedness given by Theorem 9 and, furthemore, below* s_{sob} *in* (2)*.*

Using certain Lorentz transformations, it is possible to transform the stationary solutions $u_0(x) = (x+\mathrm{i}0)^\lambda$ from Proposition 3 to time dependent solutions with singular support on noncharacteristic rays. Starting with the case $n=1$, the transformation

$$\begin{bmatrix} x \\ t \end{bmatrix} \to L \begin{bmatrix} x \\ t \end{bmatrix}, \quad L = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

keeps the quadratic form $x^2 - t^2$ invariant, while the transformation

$$\begin{bmatrix} x \\ t \end{bmatrix} \to L' \begin{bmatrix} x \\ t \end{bmatrix}, \quad L' = \begin{bmatrix} \sinh\theta & \cosh\theta \\ \cosh\theta & \sinh\theta \end{bmatrix}$$

keeps the quadratic form $t^2 - x^2$ invariant. Therefore, if $u(x,t)$ solves

$$\partial_t^2 u - \partial_x^2 u = f(u), \tag{5}$$

then $v(x,t) = u \circ L(x,t)$ and $w(x,t) = u \circ L'(x,t)$ solve

$$\partial_t^2 v - \partial_x^2 v = -f(v), \qquad \partial_t^2 w - \partial_x^2 w = +f(w),$$

respectively. In particular, if $u(x,t) \equiv u_0(x)$ is a stationary solution to (5) (with $\partial_t u(x,0) = 0$), then $v(x,t) = u_0(x\cosh\theta + t\sinh\theta)$ solves the same equation with a sign change and with initial data

$$v(x,0) = u_0(x\cosh\theta), \quad \partial_t v(x,0) = \sinh\theta\, u_0'(x\cosh\theta)$$

while $w(x,t) = u_0(x\sinh\theta + t\cosh\theta)$ solves (5) with initial data

$$w(x,0) = u_0(x\sinh\theta), \quad \partial_t v(x,0) = \cosh\theta\, u_0'(x\sinh\theta).$$

Suppose now that $u_0(x)$ has singular support equal to $x=0$. The reparametrization

$$\cosh\theta = \frac{1}{\sqrt{1-c^2}}, \quad \sinh\theta = \frac{c}{\sqrt{1-c^2}}$$

with $|c| < 1$ leads to

$$v(x,t) = u_0\Big(\frac{x+ct}{\sqrt{1-c^2}}\Big)$$

which has its singular support along the line $\{x + ct = 0, t \in \mathbb{R}\}$, that is, inside the light cone, while the singular support of the initial data is still $\{x=0\}$. Similarly, the reparametrization

$$\cosh\theta = \frac{c}{\sqrt{c^2-1}}, \quad \sinh\theta = \pm\frac{1}{\sqrt{c^2-1}}$$

with $c > 1$ leads to

$$w(x,t) = u_0\Big(\frac{\pm x+ct}{\sqrt{c^2-1}}\Big)$$

which has its singular support along the line $\{x \pm ct = 0, t \in \mathbb{R}\}$, that is, outside the light cone. In conclusion, the stationary solutions from Proposition 3 can be transformed to nonstationary solutions with singular support on any ray off the light cone.

5 Special products of distributions

The subsequent analysis requires refined estimates for products of Sobolev functions whose Fourier transform is supported in $\Gamma = [0, \infty) \subset \mathbb{R}$. For later reference we present results for $\mathbb{R}^n$.

Let Γ be a closed, acute, convex cone in $\mathbb{R}^n$ and $s \in \mathbb{R}$. Notation:

$$H^s_\Gamma(\mathbb{R}^n) = \{f \in H^s(\mathbb{R}^n) : \operatorname{supp} \widehat{f} \subset \Gamma\} = \mathcal{S}'_\Gamma(\mathbb{R}^n) \cap H^s(\mathbb{R}^n).$$

Note that the solutions given in Section 4 locally belong to $H^s_\Gamma(\mathbb{R})$ at any fixed time t, for $\Gamma = [0, \infty)$. The product of two members $f \in H^{s_1}_\Gamma(\mathbb{R}^n)$ and $g \in H^{s_2}_\Gamma(\mathbb{R}^n)$ is understood in the sense of the Fourier product.

Proposition 5. *(a) Let $s_1 \le 0$, $s_2 \le \frac{n}{2}$ and $f \in H^{s_1}_\Gamma(\mathbb{R}^n)$, $g \in H^{s_2}_\Gamma(\mathbb{R}^n)$. Then $fg \in H^\sigma_\Gamma(\mathbb{R}^n)$ for $\sigma < -\frac{n}{2} + s_1 + s_2$.*
(b) Let $s_1 \ge 0$, $s_2 \in \mathbb{R}$ and $f \in H^{s_1}_\Gamma(\mathbb{R}^n)$, $g \in H^{s_2}_\Gamma(\mathbb{R}^n)$. Then $fg \in H^\sigma_\Gamma(\mathbb{R}^n)$ for $\sigma \le s_1$, $\sigma < -\frac{n}{2} + s_2$.
In both cases, $\|fg\|_{H^\sigma} \le \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}$ for some constant $C > 0$.

Proof. (1) Assume first that Γ is the positive coordinate cone

$$\Gamma = \{\xi \in \mathbb{R}^n : \xi_i \ge 0, i = 1, \dots, n\}.$$

Write $\int_0^\xi$ for the n-dimensional integral $\int_{\xi_1} \dots \int_{\xi_n}$ etc. The proof of (a) starts with Minkowski's inequality for integrals and the observation that

$$\mathbf{1}_{[0,\xi]}(\eta) = \mathbf{1}_{[\eta,\infty)}(\xi)$$

holds for the characteristic functions of the indicated n-dimensional intervals. Thus

$$\begin{aligned}
\|\widehat{f} * \widehat{g}\|_{L^2_\sigma} &= \left(\int_0^\infty \Big| \int_0^\xi \widehat{f}(\xi - \eta)\widehat{g}(\eta)\,\mathrm{d}\eta\Big|^2 \langle\xi\rangle^{2\sigma}\,\mathrm{d}\xi\right)^{1/2} \\
&= \left(\int_0^\infty \Big| \int_0^\infty \mathbf{1}_{[0,\xi]}(\eta)\widehat{f}(\xi - \eta)\widehat{g}(\eta)\langle\xi\rangle^{\sigma}\,\mathrm{d}\eta\Big|^2 \mathrm{d}\xi\right)^{1/2} \\
&\le \int_0^\infty \left(\int_0^\infty \mathbf{1}_{[0,\eta]}(\xi)|\widehat{f}(\xi - \eta)|^2|\widehat{g}(\eta)|^2\langle\xi\rangle^{2\sigma}\,\mathrm{d}\xi\right)^{1/2} \mathrm{d}\eta \\
&= \int_0^\infty \left(\int_\eta^\infty |\widehat{f}(\xi - \eta)|^2|\widehat{g}(\eta)|^2\langle\xi\rangle^{2\sigma}\,\mathrm{d}\xi\right)^{1/2} \mathrm{d}\eta \\
&= \int_0^\infty \left(\int_0^\infty |\widehat{f}(\xi)|^2|\widehat{g}(\eta)|^2\langle\xi + \eta\rangle^{2\sigma}\,\mathrm{d}\xi\right)^{1/2} \mathrm{d}\eta.
\end{aligned}$$

For $\xi \ge 0$, $\eta \ge 0$ and $s_1 \le 0$, $\sigma - s_1 \le 0$ (which holds for the σ under consideration provided $s_2 \le \frac{n}{2}$) one has

$$\langle\xi + \eta\rangle^{2\sigma} = \langle\xi + \eta\rangle^{2s_1}\langle\xi + \eta\rangle^{2\sigma - 2s_1} \le \langle\xi\rangle^{2s_1}\langle\eta\rangle^{2\sigma - 2s_1} = \langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2 + 2\sigma - 2s_1 - 2s_2}.$$

Thus

$$\begin{aligned}
\|\widehat{f} * \widehat{g}\|_{L^2_\sigma} &\le \left(\int_0^\infty |\widehat{f}(\xi)|^2\langle\xi\rangle^{2s_1}\,\mathrm{d}\xi\right)^{1/2} \int_0^\infty |\widehat{g}(\eta)|\langle\eta\rangle^{s_2}\langle\eta\rangle^{\sigma - s_1 - s_2}\,\mathrm{d}\eta \\
&\le \|f\|_{H^{s_1}}\|g\|_{H^{s_2}} \left(\int_0^\infty \langle\eta\rangle^{2\sigma - 2s_1 - 2s_2}\,\mathrm{d}\eta\right)^{1/2}.
\end{aligned}$$

The latter integral is finite for $\sigma < -\frac{n}{2} + s_1 + s_2$.

In the situation (b), we estimate, using that $s_1 \geq 0$ and $\sigma - s_1 \leq 0$,

$$\langle \xi + \eta \rangle^{2\sigma} = \langle \xi + \eta \rangle^{2s_1} \langle \xi + \eta \rangle^{2\sigma - 2s_1} \leq \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_1} \langle \eta \rangle^{2\sigma - 2s_1} = \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2 + 2\sigma - 2s_2}.$$

This time using Hölder's inequality we get

$$\begin{aligned} \|\widehat{f} * \widehat{g}\|_{L^2_\sigma} &\leq \left(\int_0^\infty |\widehat{f}(\xi)|^2 \langle \xi \rangle^{2s_1} \, \mathrm{d}\xi \right)^{1/2} \int_0^\infty |\widehat{g}(\eta)| \langle \eta \rangle^{s_2} \langle \eta \rangle^{\sigma - s_2} \, \mathrm{d}\eta \\ &\leq \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} \left(\int_0^\infty \langle \eta \rangle^{2\sigma - 2s_2} \, \mathrm{d}\eta \right)^{1/2}, \end{aligned}$$

which is finite since $\sigma < -\frac{n}{2} + s_2$.

(2) If Γ is an arbitrary convex cone, note that if the supports of $\widehat{f}$ and $\widehat{g}$ are contained in Γ, so is the support of $\widehat{f} * \widehat{g}$. Since Γ is acute, one may assume (after rotation) that $\Gamma \subset \{(\xi', \xi_n) \in \mathbb{R}^n : \xi_n \geq \alpha |\xi'|\}$ for some $\alpha > 0$; here $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Using a dilation $\widetilde{\xi_n} = \beta \xi_n$, $\widetilde{\xi'} = \xi'$ one may make the opening angle arbitrarily small. After a further rotation, one may assume that $\Gamma \subset \{\xi \in \mathbb{R}^n : \xi_i \geq 0, i = 1, \ldots, n\}$, that is case (1). Rotations and dilations do not change $H^s(\mathbb{R}^n)$. $\square$

Remark 6. *The estimates in (a) are sharp. For example, when $n = 1$, $s \leq 0$ and $f \in H^s_\Gamma(\mathbb{R})$, (a) implies that $f^2 \in H^\sigma_\Gamma(\mathbb{R})$ for $\sigma < -\frac{1}{2} + 2s$. The bounds are realized by $f(x) = (x + \mathrm{i}0)^{-1}$, which belongs to $H^s_\Gamma(\mathbb{R})$ if and only if $s < -\frac{1}{2}$, while $f^2(x) = (x + \mathrm{i}0)^{-2}$ belongs to $H^\sigma_\Gamma(\mathbb{R})$ if and only if $\sigma < -\frac{3}{2}$ as predicted.*

The estimates in (b) are not sharp. For example, if $s = s_1 = s_2 > \frac{n}{2}$, fg is known to belong to $H^s_\Gamma(\mathbb{R}^n)$, while (b) only predicts $fg \in H^\sigma_\Gamma(\mathbb{R})$ for $\sigma < -\frac{n}{2} + s$.

On the other hand, if f and g merely belong to $H^{s_1}(\mathbb{R}^n)$ and $H^{s_2}(\mathbb{R}^n)$, then fg are only known to belong to $H^\sigma(\mathbb{R}^n)$ for $\sigma < -\frac{n}{2} + s_1 + s_2$ under the condition that $s_1 + s_2 \geq 0$ [16, Theorem 8.2.1], see also [3, Lemma 1.3] and [4, formula (1.5)].

In view of the intended application to the semilinear wave equation (1) we now assume that

$$u \in H^s_\Gamma(\mathbb{R}^n)$$

where Γ is a cone as above. Let p be a positive integer. We wish to determine ranges for σ such that $u^p \in H^\sigma_\Gamma(\mathbb{R}^n)$.

Remark 7. *(Sobolev properties of integer powers)*

The case $s \leq 0$. Here

$$u^p \in H^\sigma_\Gamma(\mathbb{R}^n) \quad \text{for} \quad \sigma < -\frac{n}{2}(p-1) + ps.$$

The case $0 < s \leq \frac{n}{2}$. Here Proposition 5(a) immediately gives that

$$u^p \in H^\sigma_\Gamma(\mathbb{R}^n) \quad \text{for} \quad \sigma < -\frac{n}{2}(p-1) + (p-1)s = (p-1)(s - \frac{n}{2}).$$

This follows by induction, using Proposition 5(a) and (b). Indeed, $u^2 \in H^\sigma_\Gamma(\mathbb{R}^n)$ for $\sigma < -\frac{n}{2} + s$ by item (b). For $p = 3$ we use item (a) with $s_1 = -\frac{n}{2} + s$, $s_2 = s$ to obtain $u^3 \in H^\sigma_\Gamma(\mathbb{R}^n)$ for $\sigma < -n + 2s$. For $p = 4$ we use again item (a) with $s_1 = -n + 2s$, $s_2 = s$ to obtain $u^4 \in H^\sigma_\Gamma(\mathbb{R}^n)$ for $\sigma < -\frac{3}{2}n + 3s$, and so on.

Note that in particular for $s = \frac{n}{2}$, $u^p \in H^\sigma_\Gamma(\mathbb{R}^n)$ for all p and $\sigma < 0$.

The case $s > \frac{n}{2}$. Here $u^p \in H^s_\Gamma(\mathbb{R}^n)$ for all p since the latter space is an algebra.

In all cases, $\|u^p\|_{H^\sigma} \le C\|u\|^p_{H^s}$ for some constant $C > 0$.

The application of Proposition 5 does not produce new estimates for the power function on $H^s_\Gamma(\mathbb{R}^n)$ for $n \ge 2$ and $s \le \frac{n}{2}$. We henceforth concentrate on the case $n = 1$. In the context of the semilinear wave equation, the question arises whether $u \in H^s_\Gamma(\mathbb{R})$ implies $u^p \in H^{s-1}_\Gamma(\mathbb{R})$, that is, whether the σ in Remark 7 can attain a value $\ge s-1$. The answer is summarized in the following remark, which is of interest when $s \le \frac{1}{2}$.

Remark 8. *Suppose that $u \in H^s_\Gamma(\mathbb{R})$, where Γ is a closed half-ray. Then $u^p \in H^{s-1}_\Gamma(\mathbb{R})$ in the following cases:*

$$\begin{aligned} &p = 2, \quad -\tfrac{1}{2} < s < \infty, \\ &p \ge 3, \quad \tfrac{1}{2} - \tfrac{1}{2p-4} < s < \infty. \end{aligned} \tag{6}$$

In all cases, $\|u^p\|_{H^{s-1}} \le C\|u\|^p_{H^s}$ for some constant $C > 0$.

6 Application to 1D-semilinear wave equations

In this section, we address wellposedness of the Cauchy problem for the semilinear wave equation

$$\partial_t^2 u - \partial_x^2 u = \pm u^p, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x) \tag{7}$$

in one space dimension; here $p \ge 2$ is a positive integer. We denote by $\Gamma \subset \mathbb{R}$ a closed half-ray which may be assumed to be the half-line $[0,\infty)$.

Theorem 9. *Assume that p and s are in the range given by (6). Let $u_0 \in H^s_\Gamma(\mathbb{R})$, $u_1 \in H^{s-1}_\Gamma(\mathbb{R})$. Then there is $T > 0$ such that problem (7) has a unique distributional solution in $\mathcal{C}([-T,T] : H^s_\Gamma(\mathbb{R})) \cap \mathcal{C}^1([-T,T] : H^{s-1}_\Gamma(\mathbb{R}))$. Further, the map $(u_0,u_1) \to u$ is locally Lipschitz continuous.*

Proof. Let $E(t,\cdot) = \mathcal{F}^{-1}\Big(\frac{\sin t|\xi|}{|\xi|}\Big)$ and

$$\Big(\mathcal{M}u\Big)(t) = \frac{\mathrm{d}}{\mathrm{d}t}E(t) * u_0 + E(t) * u_1 + \int_0^t E(t-\tau) * u^p(\tau)\,\mathrm{d}\tau.$$

The goal is to construct a fixed point $u = \mathcal{M}u$ in the ball

$$\mathcal{B}_T = \{u \in \mathcal{C}([-T,T] : H^s_\Gamma(\mathbb{R})) : \sup_{-T\le t\le T} \|u(t) - \tfrac{\mathrm{d}}{\mathrm{d}t}E(t) * u_0 - E(t) * u_1\|_{H^s(\mathbb{R})} \le 1\}$$

for small T. Note that (for $t \ge 0$)

$$\widehat{\mathcal{M}u}(\xi,t) = \cos(t|\xi|)\widehat{u_0}(\xi) + \frac{\sin t|\xi|}{|\xi|}\widehat{u_1}(\xi) + \int_0^t \frac{\sin(t-\tau)|\xi|}{|\xi|}\widehat{u}^{*p}(\xi,\tau)\,\mathrm{d}\tau$$

from where

$$\|\mathcal{M}u(t)\|_{H^s(\mathbb{R})} \le \|u_0\|_{H^s(\mathbb{R})} + \|u_1\|_{H^{s-1}(\mathbb{R})} + C\int_0^t \|u^p(\tau)\|_{H^{s-1}(\mathbb{R})}\,\mathrm{d}\tau.$$

The case $p = 2$, $s > -1/2$. Let $v \in H^s_\Gamma(\mathbb{R})$. Then $v^2 \in H^\sigma_\Gamma(\mathbb{R})$ for $\sigma < -\frac{1}{2} + 2s$, say $\sigma = -\frac{1}{2} + 2s - \varepsilon$.

We have $-\frac{1}{2}+2s-\varepsilon>s-1$ for small $\varepsilon>0$ and so

$$v^2\in H^{s-1}_\Gamma(\mathbb{R}),\qquad \|v^2\|_{H^{s-1}(\mathbb{R})}\le C\|v\|^2_{H^s(\mathbb{R})}.$$

Similarly, if $v,w\in H^\sigma_\Gamma(\mathbb{R})$ then

$$v^2-w^2\in H^{s-1}_\Gamma(\mathbb{R}),\qquad \|v^2-w^2\|_{H^{s-1}(\mathbb{R})}\le C\|v-w\|_{H^s(\mathbb{R})}\|v+w\|_{H^s(\mathbb{R})}.$$

It follows that

(a) $u\in\mathcal{C}([-T,T]:H^s_\Gamma(\mathbb{R}))\Rightarrow u^2\in\mathcal{C}([-T,T]:H^{s-1}_\Gamma(\mathbb{R}))$. Indeed, apply the estimate above to $v=u(t+h)$, $w=u(t)$.

(b) $\mathcal{M}:\mathcal{B}_T\to\mathcal{B}_T$. This follows from the estimate

$$\|\mathcal{M}u(t)\|_{H^s(\mathbb{R})}\le\|u_0\|_{H^s(\mathbb{R})}+\|u_1\|_{H^{s-1}(\mathbb{R})}+C\int_0^t\|u(\tau)\|^p_{H^s(\mathbb{R})}\,\mathrm{d}\tau.$$

(c) $\mathcal{M}$ is a contraction on $\mathcal{B}_T$ for small T. This follows from the similar estimate

$$\|\mathcal{M}u(t)-\mathcal{M}v(t)\|_{H^s(\mathbb{R})}\le C\int_0^t\|u(\tau)-v(\tau)\|_{H^s(\mathbb{R})}\sup_{0\le\tau\le T}\|u(\tau)-v(\tau)\|_{H^s(\mathbb{R})}\,\mathrm{d}\tau.$$

Existence and uniqueness follow. Also, $u=\mathcal{M}u$ and $u\in\mathcal{C}([-T,T]:H^s_\Gamma(\mathbb{R}))$ implies that $u\in\mathcal{C}^1([-T,T]:H^{s-1}_\Gamma(\mathbb{R}))$. Finally, Gronwall's inequality gives local Lipschitz continuity.

For $p\ge3$ a similar argument works using factorization of v^p-w^p and Proposition 5. □

Remark 10. *Note that the lower bound in* (6) *is smaller than* s_{sob} *in* (2) *only for* $p=2$ *and* $p=3$, *so in these cases Theorem 9 improves the results of [9, 12].*

7 Extensions to semilinear wave equations in higher dimensions

The results of the previous sections focused on new phenomena for semilinear wave equations on the real line. We now discuss their extension to higher dimensions.

In addition to $(x+\mathrm{i}0)^\lambda$, which was considered above, we introduce the radially symmetric pseudofunction $r^\lambda\in\mathcal{S}'(\mathbb{R}^n)$, defined by

$$\langle r^\lambda,\varphi\rangle=\int|x|^\lambda\varphi(x)\,\mathrm{d}x$$

for $\operatorname{Re}\lambda>-n$. It can be extended to a meromorphic $\mathcal{S}'(\mathbb{R}^n)$-valued function of $\lambda\in\mathbb{C}$ with simple poles at $\lambda=-n-2k,k\in\mathbb{N}$, [13, Section I.3.9]. Its Fourier transform is given by

$$\mathcal{F}\big(r^\lambda\big)(\rho)=\pi^{-\lambda-n/2}\frac{\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(\frac{-\lambda}{2}\right)}\rho^{-\lambda-n}$$

for $\lambda\ne-n-2k$ and $\lambda\ne2k,k\in\mathbb{N}$ [13, Section II.3.3], [40, Formula (VII,7;13)].

Remark 11. *The product of the pseudofunctions r^λ is defined as a Fourier product: First, one may extend the definition to $\lambda \in \mathbb{C}$ by taking finite parts in the poles. It was proved in [28, Satz 5] that the $\mathcal{S}'$-convolution of $\mathrm{Pf}\, r^\alpha$ and $\mathrm{Pf}\, r^\beta$ exists if and only if $\mathrm{Re}(\alpha+\beta) < -n$. This can be used to characterize the range of exponents for which the Fourier product exists. However, for the present paper, only the range $\lambda \in \mathbb{R}$, $-n < \lambda < 0$ will be needed, in which case both r^λ and $\mathcal{F}(r^\lambda)$ are locally integrable functions. We show that – in the indicated range of exponents – the Fourier product of r^λ and r^μ exists if $\lambda + \mu > -n$. Up to constant factors, the respective Fourier transforms are $\rho^{-\lambda-n}$ and $\rho^{-\mu-n}$. Take $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $(\varphi * \rho^{-\lambda-n})\rho^{-\mu-n} \in \mathcal{E}'(\mathbb{R}^n) + L^1(\mathbb{R}^n) \subset \mathcal{D}'_{L^1}(\mathbb{R}^n)$ provided $-\lambda - n - \mu - n + n - 1 < -1$. This is exactly the case when $\lambda + \mu > -n$. Thus the Fourier product of r^λ and r^μ exists in this range. A proof that $r^\lambda \cdot r^\mu = r^{\lambda+\mu}$ can be found, e.g., in [23, Example 5.4]. Incidentally, r^λ, r^μ and $r^{\lambda+\mu}$ are locally integrable functions in the range $0 > \lambda, \mu > -n$, $\lambda + \mu > -n$, and the usual product equals $r^\lambda r^\mu = r^{\lambda+\mu}$, thus coincides with the Fourier product. The same holds for integer powers $(r^\lambda)^p = r^{\lambda p}$ when $\lambda p > -n$.*

Outside the poles, the pseudofunctions r^λ satisfy

$$\Delta r^\lambda = \lambda(\lambda + n - 2) r^{\lambda - 2}.$$

In particular, when $\lambda > 2 - n$ and $p = 1 - 2/\lambda$, r^λ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $(r^\lambda)^p = r^{\lambda p}$ and it satisfies the elliptic equation

$$\Delta r^\lambda = \lambda(\lambda + n - 2)(r^\lambda)^p,$$

where the derivatives are understood in the weak sense and the pth power as the evaluation of the Nemytskii operator $L^p_{\mathrm{loc}}(\mathbb{R}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Remark 12. *The following properties are easy to show. We assume here that $\lambda < 0$ so that the pseudofunction $\rho^{-\lambda-n}$ is locally integrable.*

(a) For $\lambda < 0$, the following equivalences hold:

(1) $r^\lambda \in H^s_{\mathrm{loc}}(\mathbb{R}^n) \;\Leftrightarrow\; s < \lambda + \frac{n}{2}$,

(2) $r^\lambda \in H^s(\mathbb{R}^n) \;\Leftrightarrow\; \lambda < -\frac{n}{2}$ and $s < \lambda + \frac{n}{2}$.

(b) If $-n < \lambda < 0$, both r^λ and its Fourier transform belong to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$r^\lambda \in H^s_{\mathrm{loc}}(\mathbb{R}^n) \Leftrightarrow s < \lambda + \frac{n}{2}.$$

The conditions $\lambda - 2 > -n$ and $\lambda < 0$ can only be satisfied if $n \geq 3$. In the ranges of λ under consideration, r^λ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Proposition 13. *Let $n \geq 3$. For every $s < \frac{n}{2}$ there are $\lambda < 0$ and $p \in \mathbb{N}$ such that*

(a) the distribution $u_0(x) = r^\lambda$ belongs to $H^s_{\mathrm{loc}}(\mathbb{R})$, $\mathrm{singsupp}\, u_0 = \{0\}$, and

(b) $u(x,t) \equiv u_0(x)$ is a distributional solution to the semilinear wave equation

$$\partial_t^2 u - \Delta u = -\lambda(\lambda - 1) u^p, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = 0, \tag{8}$$

where the nonlinear term is understood in the sense of the Nemytskii operator $L^p_{\mathrm{loc}}(\mathbb{R}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Its singular support is the noncharacteristic line $\{(0,t) : t \in \mathbb{R}\}$.

Proof. It is clear that the function $u(x,t)$ satisfies the semilinear wave equation (8) when $\lambda = \frac{2}{1-p}$. As noted, r^λ belongs to $H^s_{\mathrm{loc}}(\mathbb{R})$ iff $s < \lambda + \frac{n}{2} = \frac{2}{1-p} + \frac{n}{2}$. Let $p \to \infty$ to produce the desired s. □

Remark 14. *(a) When $s > \frac{n+1}{2}$, the anomalous singular support of the solution from Propositon 13 would be ruled out by the results in [33]. Thus there is a gap between the counterexamples in Proposition 13 ($s < \frac{n}{2}$) and the nonlinear propagation results for $s > \frac{n+1}{2}$.*

(b) In order to have that $r^\lambda \in H^s(\mathbb{R}^n)$ it is necessary that $\lambda < -\frac{n}{2}$ and $s < \lambda + \frac{n}{2}$ (Remark 12). This cannot occur for $p \geq 2$ and $n \geq 2$.

It remains to be checked whether other radial solutions to the semilinear Laplace equation [10, 32] can serve for constructing anomalous solutions to (1).

Lorentz transformations can be applied, similar to the one-dimensional problem, to transform the stationary solutions $u_0(x) = r^\lambda$ from Proposition 13 to time dependent solutions with singular support on noncharacteristic rays.

Indeed, suppose that $u(x,t) \equiv u_0(x)$ is a stationary solution to

$$\partial_t^2 u - \Delta u = f(u).$$

Let $|c| < 1$ and set

$$v(x_1, \ldots, x_n, t) = u_0\Big(\frac{x_1 + ct}{\sqrt{1-c^2}}, x_2, \ldots, x_n\Big).$$

Then

$$\partial_t^2 v - \Delta v = -f(v).$$

The singular support of v is the ray $\{x_1 + ct = 0, x_2 = 0, \ldots, x_n = 0 : t \in \mathbb{R}\}$, which lies inside the light cone.

Similarly, the choice of $|c| > 1$ and

$$w(x_1, \ldots, x_n, t) = u_0\Big(\frac{\pm x_1 + ct}{\sqrt{c^2-1}}, x_2, \ldots, x_n\Big)$$

gives a solution to $\partial_t^2 w - \Delta w = f(w)$ with singular support on a ray in a coordinate plane outside the light cone. By means of a rotation of the x-coordinate system, one may produce solutions whose singular support is any ray not contained in the light cone.

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