hp finite element approximation for the fractional Laplacian on uniform and geometrically graded meshes

Heiko Gimperlein*† Ernst P. Stephan‡ Jakub Stocek§

Abstract

Solutions to the Dirichlet problem for the fractional Laplacian in a polygonal, two-dimensional domain exhibit singularities at edges and corners. This article considers their approximation by hp versions of the finite element method. On geometrically graded meshes the error in the energy norm is shown to converge exponentially fast with increasing number of degrees of freedom. Quasi-optimal convergence rates are obtained on quasi-uniform meshes. Numerical experiments confirm the theoretical results. They illustrate the expected convergence rates for the hp version on quasi-uniform meshes and the exponential convergence on geometrically graded meshes.

1 Introduction

Solutions to elliptic differential boundary value problems in polyhedral domains exhibit singularities in a neighborhood of the corners and edges. Numerical approximations by finite or boundary element methods take into account the nonsmooth behavior with local mesh refinements or higher polynomial degrees to recover optimal convergence rates. The resulting h, p and hp methods have been studied for several decades, see e.g. [43] for finite elements and [33] for boundary elements.

Nonlocal boundary value problems for fractional Laplacians [10, 31] and their numerical approximations [2, 1, 9, 4, 7, 21, 30] have attracted much recent interest. Applications of the integral fractional Laplacian $(-\Delta)^s$ arise from the pricing of stock options [50], image processing [25] and continuum mechanics [16] to the movement of biological organisms [19, 20] and the design of swarm robotic systems [17, 18].

In this article we consider the model fractional Dirichlet problem in a polygonal domain $\Omega \subset \mathbb{R}^2$ for $s \in (0,1)$,

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in}\, \Omega, \\ u &= 0 \quad \text{in}\, \Omega^c = \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \tag{1}$$

with $s \in (0,1)$. For $s = 1$ one recovers the classical Dirichlet problem for the Laplacian in Ω, for $s = \frac{1}{2}$ the hypersingular integral equation on the flat screen $\Omega \times \{0\} \subset \mathbb{R}^3$ [30].

*Engineering Mathematics, University of Innsbruck, Innsbruck, Austria, email: heiko.gimperlein@uibk.ac.at.

†Department of Mathematical, Physical and Computer Sciences, University of Parma, 43124 Parma, Italy.

‡Institute of Applied Mathematics, Leibniz University Hannover, 30167 Hannover, Germany, email: stephan@ifam.uni-hannover.de.

§British Antarctic Survey, Cambridge, CB3 0ET, UK, email: jaksto@bas.ac.uk.

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The singularities of the solution of (1) at non-smooth boundary points of the domain Ω reduce the order of convergence for the h and p versions of the finite element method. Graded meshes and hp versions are known to lead to efficient approximations for elliptic differential boundary value problems.

We exploit recent regularity results for the solution of (1) from [21] and [28] for an error analysis of the hp version on quasi-uniform and on geometrically graded meshes. Using that the solution to (1) belongs to the countably normed space $B^1_\beta(\Omega)$ (introduced by Babuška and Guo [32]), see Definition 4.3, we obtain exponential convergence for the approximation on geometrically graded meshes:

Theorem A. *Let $s \in (0,1)$, $u \in \widetilde{H}^s(\Omega)$ be a solution to* (1) *with right hand side $f \in C^\infty(\overline{\Omega})$ and u_{hp} be the hp finite element approximation on a geometrically graded mesh. Assume that for some $d \geq 1$ and all $k \geq 0$*

$$\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2(\Omega)} \leq d^{k+1} k^k.$$

Then $u \in B^1_\beta(\Omega)$ and

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \lesssim e^{-CN^{1/4}},$$

where C is independent of number of degrees of freedom N.

See [23] for related estimates on a certain class of triangular meshes, without numerical examples. We also obtain quasi-optimal convergence rates on quasi-uniform meshes:

Theorem B. *Let $s \in (0,1)$, $u \in \widetilde{H}^s(\Omega)$ be a solution to* (1) *with f sufficiently smooth, and u_{hp} be the hp finite element approximation on a quasi-uniform mesh. Then there exists $\beta \in \mathbb{N}_0$ such that*

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \leq C h^{1/2} p^{-1} \left(1 + \log(ph^{-1})\right)^\beta.$$

The assertion of Theorem A may be found in Proposition 4.5 and Corollary 4.6 below, Theorem B as Corollary 4.2.

Extensive numerical experiments in Section 8 for the h, p and hp versions confirm these theoretical results. They illustrate the exponential convergence on geometrically graded meshes and also obtain the predicted convergence rates on quasi-uniform meshes.

The p and hp approximations of elliptic equations in polyhedral domains and their optimal rates of convergence have been studied for several decades for finite element methods [6, 5, 14, 15] and boundary element methods [45, 44, 8, 35, 27, 43]. Related to this work exponential convergence of the hp version for boundary element methods has been investigated in [36, 34, 37]. For the fractional Laplacian, recent works develop the theory in 1D [3] and hp methods for the spectral and integral fractional Laplacians [7, 21, 22, 24, 39].

The article is organized as follows: After introducing the Dirichlet problem for the fractional Laplacian in Section 2, we report in Section 3 from [26, 28] detailed regularity results describing the behavior of the solution near corners and edges. In Section 4 we present our approximation results for the hp version on quasi-uniform (Subsection 4.1) and on geometrically graded meshes (Subsection 4.2). The results in Section 3 are used to show the convergence estimates for the hp version on quasi-uniform meshes in Section 5. To obtain exponentially fast convergence for the hp version on geometrically graded meshes we use the recent results by [21] together with the framework of the countably normed space B^1_β [32]. In Section 6 we present our proof of the exponential convergence based on separate treatment of the corner element, the edge elements and the elements away from the boundary on a tensor product mesh. In Section 7 the implementation of the methods is described, and in Section 8 numerical results are presented.

Notation: We write $f \lesssim g$ provided there exists a constant C such that $f \leq Cg$. If the constant C is allowed to depend on a parameter σ, we write $f \lesssim_\sigma g$.

2 Setup

We recall basic definitions and properties related to Sobolev spaces of non-integer order and to the fractional Laplacian. For further details we refer to [13].
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and for $s \in \mathbb{N}_0$, $H^s(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose distributional derivatives of order s belong to $L^2(\Omega)$. For $s \in (0,\infty)$, we write $m = \lfloor s \rfloor$ and $\sigma = s - m$ and define the Sobolev space $H^s(\Omega)$ as

$$H^s(\Omega) = \{v \in H^m(\Omega) : |\partial^\alpha v|_{H^\sigma(\Omega)} < \infty \ \ \forall |\alpha| = m\} \ .$$

Here $|\cdot|_{H^\sigma(\Omega)}$ is the Aronszajn-Slobodeckij seminorm

$$|v|^2_{H^\sigma(\Omega)} = \iint_{\Omega\times\Omega} \frac{(v(x)-v(y))^2}{|x-y|^{n+2\sigma}}\, \mathrm{d}y\, \mathrm{d}x.$$

$H^s(\Omega)$ is a Hilbert space endowed with the norm

$$\|v\|^2_{H^s(\Omega)} = \|v\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} |\partial^\alpha v|^2_{H^\sigma(\Omega)}.$$

Particularly relevant for this article is the space

$$\widetilde{H}^s(\Omega) = \{v \in H^s(\mathbb{R}^n) : \operatorname{supp} v \subset \overline{\Omega}\}$$

of distributions whose extension by 0 belongs to $H^s(\mathbb{R}^n)$.

We recall that when Ω is Lipschitz and $\frac{1}{2} \neq s \in (0,1)$, $\widetilde{H}^s(\Omega)$ coincides with the space $H^s_0(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the H^s norm. Moreover, for $s \in (0,\frac{1}{2})$, $\widetilde{H}^s(\Omega) = H^s(\Omega) = H^s_0(\Omega)$. All three spaces differ when $s = \frac{1}{2}$.

For negative s the Sobolev spaces are defined by duality.

The following result will be useful to obtain estimates in Ω from estimates on subdomains. It is stated in Theorem A.10 in [46] and originally in [41].

Lemma 2.1. *Let Ω, Ω_j $(j = 1,\dots,k)$ be Lipschitz domains with $\overline{\Omega} = \bigcup_{j=1}^k \overline{\Omega}_j$. Then for all $s \in [-1,1]$ and all $v \in \widetilde{H}^s(\Omega)$*

$$\|v\|^2_{\widetilde{H}^s(\Omega)} \leq \sum_{j=1}^k \|v\|^2_{\widetilde{H}^s(\Omega_j)} \ . \tag{2}$$

For $s \in (0,1)$, we define the fractional Laplacian of a Schwartz function u on $\mathbb{R}^n$ by

$$(-\Delta)^s u(x) = c_{n,s}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, \mathrm{d}y = c_{n,s} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus\overline{\mathcal{B}}_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, \mathrm{d}y \ , \tag{3}$$

where $P.V.$ denotes the Cauchy principal value and $\mathcal{B}_\varepsilon(x)$ the n-dimensional ball of radius $\varepsilon > 0$ centered at X. The normalization constant $c_{n,s}$ is defined in terms of Γ functions:

$$c_{n,s} = \frac{2^{2s} s \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}} \Gamma(1-s)}.$$

Equivalently, the fractional Laplacian may be defined in terms of the Fourier transform on $\mathbb{R}^n$ as $\mathcal{F}((-\Delta)^s u) = |\xi|^{2s}\mathcal{F}u$. This expression extends $(-\Delta)^s$ to an unbounded operator on $L^2(\mathbb{R}^n)$. It also shows that $(-\Delta)^s$ is an operator of order $2s$ and that for $s = 1$ one recovers the ordinary Laplace operator.

2.1 Dirichlet problem for fractional Laplacian in a domain

In this subsection we recall the fractional Laplace problem with Dirichlet boundary conditions and the corresponding weak formulation. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $f \in L^2(\Omega)$. The weak formulation of the Dirichlet problem for the fractional Laplacian (1) involves the bilinear form a on $\widetilde{H}^s(\Omega)$,

$$\mathsf{a}(u,v) = \frac{c_{n,s}}{2} \iint_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \,\mathrm{d}y\,\mathrm{d}x \ , \tag{4}$$

where $D = (\mathbb{R}^n \times \Omega) \cup (\Omega \times \mathbb{R}^n)$.

Note that formally

$$\mathsf{a}(u,v) = \langle (-\Delta)^s u, v \rangle_{H^s(\mathbb{R}^n)} - \iint_{\Omega^c \times \Omega^c} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \,\mathrm{d}y\,\mathrm{d}x,$$

when $u, v \in H^s(\mathbb{R}^n)$, and the second term vanishes on $\widetilde{H}^s(\Omega)$. The weak form of (1) therefore reads as follows:

Find $u \in \widetilde{H}^s(\Omega)$ such that

$$\mathsf{a}(u,v) = \langle f, v \rangle \tag{5}$$

for all $v \in \widetilde{H}^s(\Omega)$.

One verifies that a is continuous and elliptic in $\widetilde{H}^s(\Omega)$: There exist $C_\mathsf{a}, \alpha > 0$ with

$$\mathsf{a}(u,v) \leq C_\mathsf{a} \|u\|_{\widetilde{H}^s(\Omega)} \|v\|_{\widetilde{H}^s(\Omega)}, \quad \mathsf{a}(u,u) \geq \alpha \|u\|^2_{\widetilde{H}^s(\Omega)} \ .$$

By Lax-Milgram, the weak form (5) admits a unique solution, and the solution operator $f \mapsto u$ extends to an isomorphism from $H^{-s}(\Omega)$ to $\widetilde{H}^s(\Omega)$.

3 Regularity theory

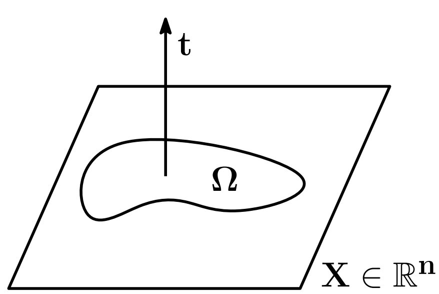


Figure 1: Geometry of the extension problem in the upper half space.

In this section we summarize the conclusions from [26, 28]: The solution to the fractional Laplace equation with Dirichlet boundary conditions near the boundary admits a decomposition into edge, corner and edge-corner singularities, plus a remainder which is smooth. Such a decomposition allows us to derive quasi-optimal convergence rates for the *hp* version on quasi-uniform meshes.

Following [10] for $\mathbb{R}^n$, resp. [26] for Ω, we introduce a boundary value problem for a degenerate partial differential operator in the half space $\mathbb{R}^n \times \mathbb{R}_+$:, which is equivalent to (1):

$$L_s U(X,t) := t^{-\alpha} \nabla \cdot (t^\alpha \nabla U(X,t)) = \partial_t^2 U + \frac{1-2s}{t} \partial_t U + \Delta_X u, \tag{6}$$

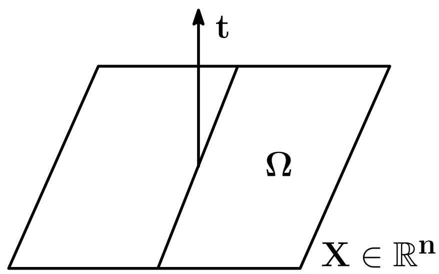


Figure 2: Model geometry for straight boundary.

with $\alpha = 1 - 2s$. Here $(X,t) \in \mathbb{R}^n \times \mathbb{R}_+$. Including the boundary conditions the model problem (1) is equivalent to:

$$\begin{aligned} L_s U(x,y,t) &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ U(x,y,0) &= 0 \quad \text{in } \Omega^C \times \{0\} \\ -\lim_{t\to 0^+} t^\alpha \partial_t U(x,y,t) &= f \quad \text{in } \Omega \times \{0\}. \end{aligned} \tag{7}$$

See Figure 1 for a depiction of the geometry of this boundary value problem when $n = 2$.

We first consider the behaviour of u near a point on an edge of $\partial\Omega$ away from any corner. The model problem for edge singularities is given by the half space $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ with coordinates $X = (x,y)$, depicted in Figure 2. For the behaviour of solutions near a general smooth boundary see for example [26, 31]. For $s = 1/2$ see also [48].

We introduce cylindrical coordinates $(x,y,t) = (\rho\sin(\theta), y, \rho\cos(\theta))$ in the half-space $\mathbb{R}^n \times \mathbb{R}_+$. Near a point on $\partial\Omega$, the solution U to the extended problem (7) then admits an expansion of the form

$$U(\rho,\theta,y) \sim \sum_j \rho^{\nu_j} \hat{u}_{e,j}(\theta,y), \tag{8}$$

up to a smooth remainder [26, 28]. Here for every y, $\hat{u}_{e,j}(\theta,y)$ is a generalized eigenfunction for a spectral problem for the operator

$$\mathcal{P}_s = \partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta$$

on the halfcircle $S^1_+ \simeq (0,\pi)$. The spectral problem is given by

$$\begin{aligned} \mathcal{P}_s\varphi &= -\lambda^2\varphi && \text{for } \theta \in (0,\pi), \\ \lim_{\theta\to 0^+} \theta^\alpha \partial_\theta \varphi &= 0 && \text{for } \theta = 0, \\ \varphi &= 0 && \text{for } \theta = \pi, \end{aligned} \tag{9}$$

where the singular exponents ν and the eigenvalues $-\lambda^2$ are related by $\lambda^2 = \nu^2 + (1-2s)\nu$. One can explicitly compute $\hat{u}_{e,j}(\theta,y) = c(y)P^s_j(\cos(\theta))\sin^s(\omega)$, where P^s_j denotes the associated Legendre function of the first kind, and one obtains $\nu_j = s + j$.

Similarly, the model geometry to describe the solution U to the extended problem (7) near a corner point with opening angle χ is given by $\Omega = \{(r,\varphi,\theta = \frac{\pi}{2}) : r > 0, \varphi \in (0,\chi)\}$, see Figure 3. The solution admits an expansion in spherical coordinates (r,θ,φ) of the form

$$U(r,\theta,\varphi) \sim \sum_j \sum_{k=0}^{N_j} r^{\lambda_j} \log(r)^k \hat{u}_{c,jk}(\theta,\varphi), \tag{10}$$

up to a smooth remainder. Here $\hat{u}_{c,jk}$ are the generalized eigenfunctions for a spectral problem for the operator

$$\mathcal{D}_{\theta,\varphi} U := \Delta_{\theta,\varphi} U - (1-2s)\tan(\theta)\partial_\theta U. \tag{11}$$

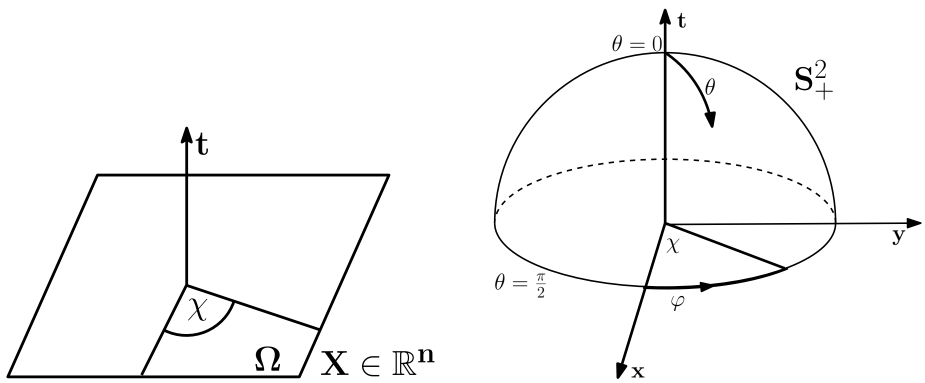


Figure 3: Model geometry for a corner with the opening angle χ.

on S_+^2 with mixed boundary conditions:

$$\mathcal{D}_{\theta,\varphi}\,\hat{\mathrm{u}} = -\mu^2\,\hat{\mathrm{u}} \quad \text{in } \mathbb{S}^2 \cap \mathbb{R}^3_+ = S^2_+$$
$$\lim_{\theta\to\frac{\pi}{2}} \left(\frac{\pi}{2}-\theta\right)^{1-2s} \partial_\theta\,\hat{\mathrm{u}} = 0, \quad \text{for } \varphi \in (0,\chi), \tag{12}$$
$$\hat{\mathrm{u}} = 0 \quad \text{for } \varphi \notin (0,\chi),\ \theta = \pi/2.$$

The relation between the eigenvalues μ and the corner exponents λ is given by

$$\mu^2 = \lambda^2 + (2-2s)\lambda. \tag{13}$$

The eigenfunctions $\hat{u}_{c,jk}$ of (12) are not smooth, but exhibit singularities as $\theta \to \frac{\pi}{2}$, because of the mixed boundary conditions and the singular behavior of the first-order term $(1-2s)\tan(\theta)\partial_\theta$ in the operator $\mathcal{D}_{\theta,\varphi}$. We first discuss the local behavior near $(\theta,\varphi) = (\frac{\pi}{2},0)$, where the boundary conditions jump. The discussion equally applies to the local behavior near $(\theta,\varphi) = (\frac{\pi}{2},\chi)$. For $\varphi \notin \{0,\chi\}$, the trace of $\hat{u}_j$ on the equator $\theta = \frac{\pi}{2}$ is smooth, and the corresponding singularities of the solution U to the extended problem (7) are not relevant to the solution u of the fractional boundary value problem.

To understand the behavior around $(\theta,\varphi) = (\frac{\pi}{2},0)$, we introduce polar coordinates (ϱ,ω),

$$\frac{\pi}{2} - \theta = \varrho\sin(\omega),\ \ \varphi = \varrho\cos(\omega)\ , \tag{14}$$

The operator $\Delta_{\theta,\varphi}$ coincides with the operator L_s in the half-space from (6) to leading order, i.e. up to terms which vanish at $\varrho = 0$ and do not affect the singular exponents, see Figure 4 for illustration. Lower-order terms are due to the curvature of S^2_+. We conclude that near $\varrho = 0$ also the eigenfunctions $\hat{u}_{c,jk}$ of $\mathcal{D}_{\theta,\varphi}$ admit an expansion with the exponents $\nu_k^{M,s} = s + k$, so that

$$\hat{u}_{c,jk} \sim \sum_{l,m} \hat{u}_{c,jk,lm}\varrho^{s+l}\log(\varrho)^m P_l^s(\cos(\omega))\sin^s(\omega)\ . \tag{15}$$

The solution to the original fractional problem (1) is given by the trace of the solution U to (7) at $\theta = \frac{\pi}{2}$, $u(r,\varphi) = U(r,\frac{\pi}{2},\varphi)$. When φ is strictly between 0 and χ, the trace of the eigenfunction $\hat{u}_{c,jk}(\frac{\pi}{2},\varphi)$ is smooth, and the asymptotic expansion (10) takes the form

$$u(r,\varphi) \sim \sum_j \sum_{k=0}^{N_j} r^{\lambda_j}\log(r)^k\hat{u}_{c,jk}(\tfrac{\pi}{2},\varphi), \tag{16}$$

with λ_j from (13). We now translate back from (r,φ) to (x,y) in this region, using $r = \sqrt{x^2+y^2}$ and $\varphi = \tan^{-1}(y/x)$, the latter of which is here a smooth function of x and y.

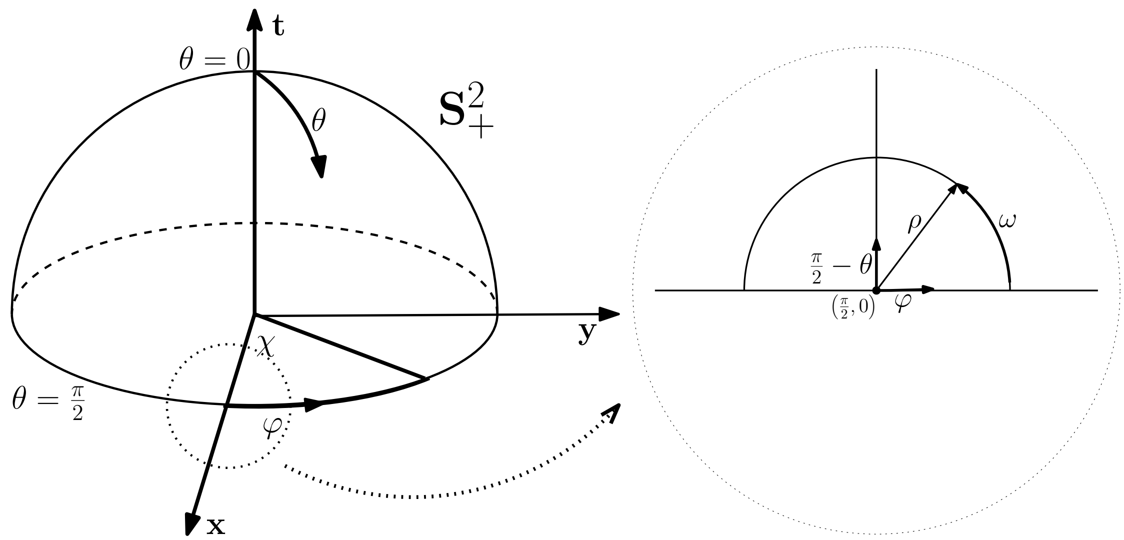


Figure 4: Model geometry near the change of boundary conditions. Highlighted region near $(\theta,\varphi)=(\pi/2,0)$ (left) and zoomed in (right) with local polar coordinates (ρ,ω).

Therefore with $\hat{U}_{c,jk}(x,y)$ obtained from $\hat{u}_{c,jk}$ by change of coordinates,

$$u(x,y)\sim\sum_j\sum_{k=0}^{N_j}\hat{U}_{c,jk}(x,y)\;(x^2+y^2)^{\lambda_j/2}\log(x^2+y^2)^k\;. \tag{17}$$

The behavior at the boundaries $\varphi=0$ and $\varphi=\chi$ is analogous, and we only discuss the edge-vertex singularities for $\varphi=0$. There

$$\hat{u}_{c,jk}\sim\sum_l\sum_{m=1}^{M_l}\hat{u}_{jk,lm}\varrho^{s+l}\log(\varrho)^m P_l^s(\cos(\omega))\sin^s(\omega)\;.$$

The trace u at $\theta=\frac{\pi}{2}$ corresponds to $\varrho\sim\varphi$ and $\omega=0$ (see Figure 4), so that

$$\hat{u}_{c,jk}(\tfrac{\pi}{2},\varphi)\sim\textstyle\sum_l\sum_{m=1}^{M_l}\widetilde{u}_{jk,lm}\varphi^{s+l}\log(\varphi)^m$$

near $\varphi=0$. Here the $\widetilde{u}_{jk,lm}$ are the expansion coefficients of $\hat{u}_{c,jk}(\frac{\pi}{2},\varphi)$ at $\varphi=0$. Therefore in this region

$$u(r,\varphi)\sim\sum_{j,k,l,m}\widetilde{u}_{jk,lm}r^{\lambda_j}\log(r)^k\varphi^{s+l}\log(\varphi)^m\;. \tag{18}$$

Translating from spherical coordinates (r,φ,θ) at $\theta=\frac{\pi}{2}$ back to $(x,y)=(r\cos(\varphi)\sin(\theta),r\cos(\varphi)\cos(\theta))$, we expand $\sin(\varphi)$ and $\cos(\varphi)$ into a Taylor series at $\varphi=0$ to see

$$y\sim r\varphi+r\sum_{i>0}\frac{(-1)^i}{(2i+1)!}\varphi^{2i+1},\quad x\sim r+r\sum_{i>0}\frac{(-1)^i}{(2i)!}\varphi^{2i}\;,$$

or up to higher order terms

$$r\sim x,\quad \varphi\sim y/x\;.$$

We conclude that in Cartesian coordinates near $\varphi=0$, i.e. $y=0$, u has an expansion of the form

$$u(x,y)\sim\sum_{j,k,l,m}v_{jk,lm}x^{\lambda_j-s-l}y^{s+l}\log(x)^k\log(y)^m\;. \tag{19}$$

Analogously, we can proceed for edge singularities. We summarize this discussion in the following theorem, which is shown in [28]. Related results for the weakly singular or hypersingular operators can be found in [42, 40].

Theorem 3.1 ([28]). *Let $u \in \widetilde{H}^s(\Omega)$ be the solution to (1) for $f \in C^\infty(\overline{\Omega})$ in a polygonal domain $\Omega \subset \mathbb{R}^2$. Then in polar coordinates in a neighborhood of each vertex of Ω*

$$u(r,\varphi) \sim v_0 + \sum_{j,k,l,m} \widetilde{u}_{jk,lm} r^{\lambda_j} \log(r)^k \varphi^{s+l} \log(\varphi)^m \ , \tag{20}$$

where v_0 is a sufficiently smooth remainder, $\widetilde{u}_{jk,lm}$ from (18) and λ_j as in (13). In Cartesian coordinates, this corresponds to a vertex singularity

$$u(x,y) \sim v_0 + \sum_j \sum_{k=0}^{N_j} \hat{U}_{jk}(x,y)\ (x^2+y^2)^{\lambda_j/2} \log(x^2+y^2)^k \tag{21}$$

away from the edge $\varphi = 0$, i.e. $y = 0$, where v_0 is a sufficiently smooth remainder.
The edge-vertex singularity near $\varphi = 0$, i.e. $y = 0$, is of the form

$$u(x,y) \sim v_0 + \sum_{j,k,l,m} v_{jk,lm} x^{\lambda_j - s - l} y^{s+l} \log(x)^k \log(y)^m, \tag{22}$$

where v_0 is a sufficiently smooth remainder and $v_{jk,lm}$ from (19).
Away from the vertex, in a neighborhood of the edge at $\varphi = 0$, i.e. $y = 0$,

$$u(x,y) \sim v_0 + \sum_{j,k} \bar{u}_{jk}(x) y^{s+j} \log(y)^k \ , \tag{23}$$

where v_0 is a sufficiently smooth remainder and $\bar{u}_{jk}$ is obtained from (8).

The logarithms in the expansions only occur for integer singular exponents or for singular exponents corresponding to multiple eigenvalues. Note that the smooth remainder v_0 represents a different function in each of the expansions.

The explicit asymptotics in Theorem 3.1 provide a detailed description of the solution near edges and corners, going beyond e.g. the weighted Sobolev estimates available in [21].

The above theorem addressed the local behavior of the solution near an edge or corner point. The global structure of the solution in a polygonal domain Ω then follows from Theorem 3.1 by combining these local descriptions, yielding Theorem 3.2 below.

To state the result, let V, E denote the sets of vertices and edges, respectively, of Ω. Denote by $E(v)$ the set of edges connected to the vertex $v \in V$.

Theorem 3.2 ([28]). *For sufficiently smooth f the solution of (1) has the form:*

$$u = u_{\text{reg}} + \sum_{e \in E} u^e + \sum_{v \in V} u^v + \sum_{v \in V} \sum_{e \in E(v)} u^{ev}, \tag{24}$$

where using local coordinate systems (r_v, θ_v) and (x_{e_1}, x_{e_2}) with origin at v, there exists the following representation:

1. *The regular part $u_{\text{reg}} \in H^k$ for some $k > s$.*
2. *The edge singularities u^e of the form*

$$u^e = \sum_{j=0}^{m_e - 1} \left(\sum_{k=0}^{k_j^e} b_{jk}^e (x_{e1}) \left|\log x_{e2}\right|^k \right) x_{e2}^{s+j} \chi_1^e (x_{e1}) \chi_2^e (x_{e2}), \tag{25}$$

where m_e, k_j^e are integers. Here, χ_1^e, χ_2^e are C^∞ cut-off functions where $\chi_1^e = 1$ away from the endpoints and zero at the end points. Furthermore, $\chi_2^e = 1$ for $0 \leq x_{e_2} \leq \delta_e$ and zero for $x_{e_2} \geq 2\delta_e$ for $\delta_e \in (0, 1/2)$. The functions $b_{jk}^e \chi_1^e \in H^m(e)$ for arbitrarily large m.

3. *The vertex singularities* u^v *of the form*

$$u^v = \chi^v(r_v) \sum_{i=1}^{n_v} \sum_{k=0}^{q_i^v} B_{ik}^v \left|\log r_v\right|^k r_v^{\lambda_i^v} w_{ik}^v(\theta_v), \tag{26}$$

where $\lambda_{i+1}^v \geq \lambda_i^v > \max\{0, s - \frac{1}{2}\}$ *as in* (13), $n_v, q_i^v \geq 0$ *are integers, and* B_{it}^v *are real numbers. The* C^∞ *cut-off function* $\chi^v = 1$ *for* $0 \leq r_v \leq \tau_v$ *and* $\chi^v = 0$ *for* $r_v \geq 2\tau_v$ *with* $\tau_v \in \left(0, \frac{1}{2}\right)$. *The functions* $w_{ik}^v \in H^q(0, \omega_v)$ *for arbitrarily large* q. *Further* ω_v *denotes the interior angle between the edges at* v.

4. *The edge-vertex singularities* u^{ev} *of the form*

$$u^{ev} = u_1^{ev} + u_2^{ev}, \tag{27}$$

where

$$u_1^{ev} = \sum_{j=0}^{m_e-1} \sum_{i=1}^{n_v} \left(\sum_{k=0}^{k_j^e} \sum_{t=0}^{q_i^v} \sum_{l=0}^{k} B_{ijlt\sigma}^{ev} \left|\log x_{e1}\right|^{k+t-l} \left|\log x_{e2}\right|^l \right) x_{e1}^{\lambda_i^v - s - j} x_{e2}^{s+j} \chi^v(r_v) \chi^{ev}(\theta_v), \tag{28}$$

$$u_2^{ev} = \sum_{j=0}^{m_e-1} \sum_{k=0}^{k_j^e} B_{jk}^{ev}(r_v) \left|\log x_{e2}\right|^k x_{e2}^{s+j} \chi^v(r_v) \chi^{ev}(\theta_v), \tag{29}$$

and

$$B_{jk}^{ev}(r_v) = \sum_{l=0}^{k} B_{jkl}^{ev}(r_v) \left|\log r_v\right|^l. \tag{30}$$

Here $q_i^v, k_j^e, \lambda_i^v, \chi^v$ *are as above,* $B_{ijlt\sigma}^{ev}$ *are real numbers, and* χ^{ev} *is a* C^∞ *cut-off function with* $\chi^{ev} = 1$ *for* $0 \leq \theta_v \leq \beta_v$ *and* $\chi^{ev} = 0$ *for* $\frac{3}{2}\beta_v \leq \theta_v \leq \omega_v$ *with* $\beta_v \in (0, \min\{\omega_v/2, \pi/8\}]$. *The functions* $B_{j\sigma l}^{ev}$ *can be chosen so that*

$$B_{j\sigma}^{ev}(r_v) \chi^v(r_v) \chi^{ev}(\theta_v) = \chi_{j\sigma}(x_{e1}, x_{e2}) \chi_2^e(x_{e2}), \tag{31}$$

where the extension of $\chi_{j\sigma}$ *by zero on* $\mathbb{R}_+^2$ *lies in* $H^m(\mathbb{R}_+^2)$ *for* m *arbitrarily large.*

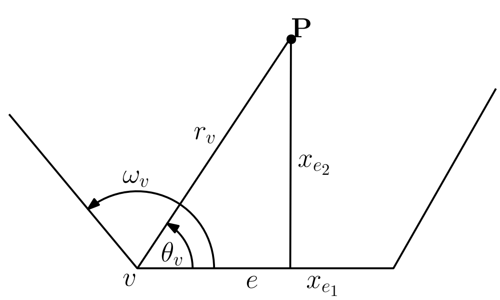


Figure 5: Diagram of the local coordinates near a vertex v and edge e.

4 Approximation results

For the numerical approximation, without loss of generality we assume that Ω has a polygonal boundary. Let $\mathcal{T}_h$ be a family of triangulations of Ω and $\widetilde{\mathbb{V}}_{hp} \subset \widetilde{H}^s(\Omega)$, the associated space of

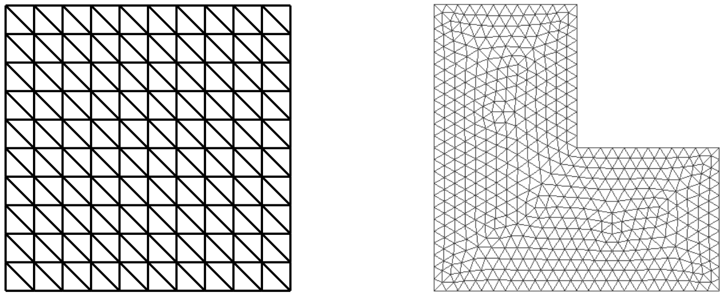


Figure 6: Examples of quasi–uniform meshes for a square and an L–shaped domain.

continuous piecewise polynomial functions of degree p on $\mathcal{T}_h$ vanishing at the boundary, with $p \geq 1$.

The discretized problem is solved on quasi-uniform triangulations $\mathcal{T}_h$ of Ω as in Figure 6.

We also consider geometrically graded quadrilateral meshes $\mathcal{Q}_h$ on Ω. To define them on an interval $\Omega = [0, 2]$ and with a refinement parameter $\sigma \in (0, 1/2]$, in the subintervall $[0, 1]$ we let $x_0 = 0$,

$$x_k = \sigma^{N+1-k} \tag{32}$$

for $k = 1, \ldots, N$, and we specify corresponding nodes in $[1, 2]$ by symmetry. For the hp version the polynomial degree p increases linearly from $\partial\Omega$: $p = \mu k$ in $[x_k, x_{k+1}]$ for a given $\mu > 0$. We denote the corresponding space of piecewise polynomial functions by $\mathbb{S}_{h_l}$. The nodes of the geometrically graded mesh on a square are again given by (x_k, y_ℓ), for $k, \ell = 1, \ldots, N$, and by symmetry extended to the whole square.
Examples of geometrically graded meshes with $\sigma = 0.5$, respectively $\sigma = 0.17$, are depicted in Figure 7 for a square and in Figure 8 for an L-shaped domain. For a polygonal domain one defines a geometrically graded mesh by including a small number of triangles in the interior and near the edges, but not in the corners, see Figure 9. More precisely, first note that every polygonal domain can be decomposed into triangles. We divide each of these triangles F into three parallelograms and three triangles where each parallelogram lies in a corner of F and each triangle lies at an edge of F away from the corners. By linear transformations we can transform the parallelograms on a reference square $Q = [0, 1]^2$ such that the vertices of F are transformed to $(0, 0)$. The triangles can be transformed by a linear transformation $\tilde{\varphi}_i$ on the reference triangle $\tilde{Q} = \{(x, y) \in Q \,|\, y \leq x\}$ such that the corner point of the triangle in the interior of the face F is transformed to $(1, 1)$ of the reference triangle. The geometric mesh and appropriate polynomial function spaces are defined on the reference element Q. Analogously the geometric mesh can be defined on the reference triangle $\tilde{Q}$ (see Figure 9). Via the linear transformations above, the geometric mesh is also defined on the polyhedron. The approximation on the reference square is the more interesting case because it handles the corner-edge singularities. Therefore we deal in this paper only with the approximation on the reference square.

4.1 Approximation results on quasi-uniform meshes

The discretized problem on a quasi-uniform mesh is given in terms of the bilinear form in (4):

Find $u_{hp} \in \widetilde{\mathbb{V}}_{hp}$, such that for all $v_{hp} \in \widetilde{\mathbb{V}}_{hp}$

$$\mathsf{a}(u_{hp}, v_{hp}) = (f, v_{hp})_{L^2(\Omega)} \ .$$

By coercivity, there exists a unique solution u_{hp}, the Galerkin approximation to u.

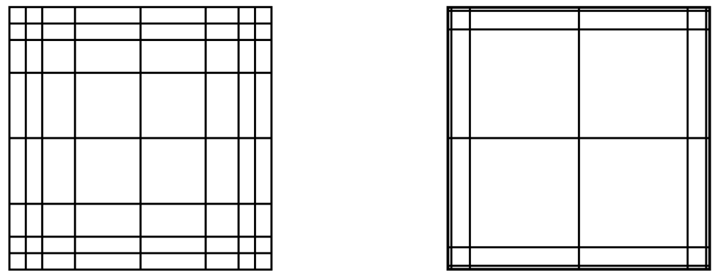


Figure 7: Examples of geometrically graded meshes with $\sigma = 0.5$ and 0.17 for the square.

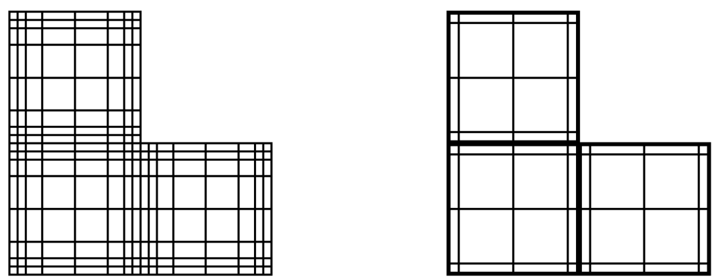


Figure 8: Examples of geometrically graded meshes with $\sigma = 0.5$ and 0.17 for an L–shaped domain.

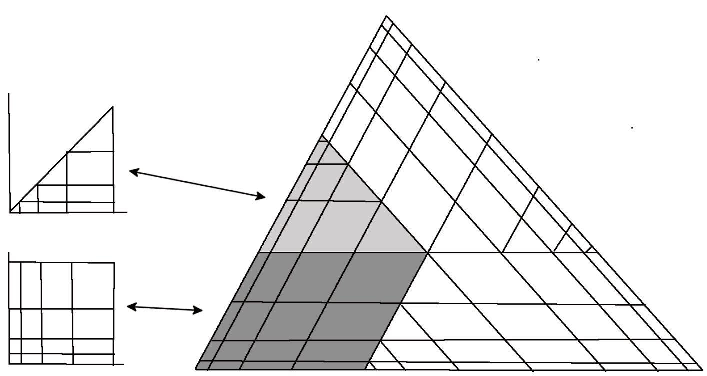


Figure 9: Graded mesh on triangle.

We adapt the analysis of Bespalov and Heuer [8], for the singular expansion from Theorem 3.2 above. Details are given in Section 5.

Theorem 4.1. *Let $u \in \widetilde{H}^s(\Omega)$ be a solution to* (1) *and $\Pi_{hp}u$ the best approximation in $\widetilde{\mathbb{V}}_{hp}$ on a quasi-uniform mesh $\mathcal{T}_h$ in the $\widetilde{H}^s(\Omega)$ norm. Then there exists $\beta \in \mathbb{N}_0$ such that*

$$\|u - \Pi_{hp}u\|_{\widetilde{H}^s(\Omega)} \leq Ch^{1/2}p^{-1}\left(1 + \log(ph^{-1})\right)^{\beta}.$$

By Cea's Lemma this theorem implies a corresponding estimate for the error of the finite element solution:

Corollary 4.2. *Let $u \in \widetilde{H}^s(\Omega)$ be a solution to* (1) *and $u_{hp} \in \widetilde{\mathbb{V}}_{hp}$ the hp finite element approximation on a quasi-uniform mesh $\mathcal{T}_h$. Then there exists $\beta \in \mathbb{N}_0$ such that*

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \leq Ch^{1/2}p^{-1}\left(1 + \log(ph^{-1})\right)^{\beta}.$$

4.2 Approximation results on geometrically graded meshes

Let $\mathcal{Q}_h$ be a mesh for Ω, consisting of M rectangular elements τ_m and R boundary edges $e_r \in \partial\mathcal{Q}_h$. Let $\chi_m : Q \to \tau_m$ be an affine transformation from a reference element $Q = [0,1]^2$ to the element $\tau_m \in \mathcal{Q}_h$ and $\chi_r : e \to e_r$ be an affine transformation from a reference edge $e = [0,1]$ to the boundary edge $e_r \in \partial\mathcal{Q}_h$. Furthermore, let $\tilde{\varphi}^m_{k,l}|_{\tau_m}(x) := \varphi_{k,l}(\chi_m^{-1}(x))$ and let $\mathbf{p} = (p_1,\ p_2,\ \ldots,\ p_M)$ be a vector of polynomial pairs $p_m = (p_{m,x_1},\ p_{m,x_2})$ associated with an element τ_m for all elements in $\mathcal{Q}_h$. The *hp* version on a geometrically graded mesh uses the mesh and degree distribution described at the beginning of this section.

The *hp* version on a geometrically graded mesh $\mathcal{Q}_h$ then uses the finite element subspace $\widetilde{\mathbb{S}}_{h\mathbf{p}} \subset \widetilde{H}^s(\Omega)$ given by

$$\widetilde{\mathbb{S}}_{h\mathbf{p}} = \{u \in \widetilde{H}^s(\Omega) : u \text{ continuous},\ u|_{\tau_m} \in \mathcal{P}^{p_{m,x_1}, p_{m,x_2}},\ \forall \tau_m \in \mathcal{T}_h\}.$$

The discretization of the weak formulation of (1) is then given in terms of the bilinear form in (4):

Find $u_{hp} \in \widetilde{\mathbb{S}}_{h\mathbf{p}}$, such that for all $v_{hp} \in \widetilde{\mathbb{S}}_{h\mathbf{p}}$

$$\mathsf{a}(u_{hp}, v_{hp}) = (f, v_{hp})_{L^2(\Omega)}\ .$$

By coercivity, there exists a unique solution $u_{hp} \in \widetilde{\mathbb{S}}_{h\mathbf{p}}$, the Galerkin approximation to u.

In order to state Theorem 4.4 below, we next introduce a scale of weighted Sobolev spaces and an associated countably normed space. On the reference element $Q = [0,1]^2$ the weight function $\Phi_{\beta,\alpha,1}$ is given by

$$\Phi_{\beta,(\alpha_1,\alpha_2),1} = \begin{cases} x^{\beta+\alpha_1-1} & \text{for } \alpha_1 \geq 1, \alpha_2 = 0 \\ x^{\beta+\alpha_1-1}y^{\alpha_2} + x_1^{\alpha}y^{\beta+\alpha_2-1} & \text{for } \alpha_1 \geq 1, \alpha_2 \geq 1 \\ y^{\beta+\alpha_2-1} & \text{for } \alpha_1 = 0, \alpha_2 \geq 1. \end{cases} \tag{33}$$

Definition 4.3. a) Let $k \geq 1$. A function $u \in L^2(\Omega)$ belongs to the weighted Sobolev space $H^{k,1}_{\beta}(\Omega)$ if $\Phi_{\beta,\alpha,1}\partial^{\alpha}u \in L^2(Q)$ for all $1 \leq |\alpha| \leq k$.
b) We say that u belongs to the countably normed space $B^1_{\beta}(\Omega)$ if $u \in \bigcap_{k\geq 1} H^{k,1}_{\beta}(\Omega)$ and there exist $C, d \geq 1$ such that for all $k \geq 1$ and all $|\alpha| = k$:

$$\|\Phi_{\beta,\alpha,1}\partial^{\alpha}u\|_{L^2(\Omega)} \leq Cd^{k-1}(k-1)!$$

Theorem 4.4. *Let Ω be a polygonal domain, $U \in B^1_\beta(\Omega)$ with $\beta \in (0,1)$ and $\Pi_{h\mathbf{p}} U \in \widetilde{\mathbb{S}}_{h\mathbf{p}}$ the best approximation to U with respect to the $\widetilde{H}^s(\Omega)$ norm. Then*

$$\|U - \Pi_{h\mathbf{p}} U\|_{\widetilde{H}^s(\Omega)} \lesssim e^{-CN^{1/4}},$$

where C is independent of N.

In the approximation arguments we regularly restrict to a rectangular reference element $Q = [0,1]^2$, as described in the introduction to this Section 4. From Figure 9, the discretization in a general polygonal domain requires additional triangles in the interior and at the edges, away from the vertices. The corresponding approximation properties in these elements are easier, as they do not involve the edge-vertex behavior; see the discussion at the beginning around Figure 9 above.

We use Theorem 2.1 from [21] to show:

Proposition 4.5. *Let $u \in \widetilde{H}^s(\Omega)$ to (1) with right hand side $f \in C^\infty(\overline{\Omega})$. Assume that for some $d \geq 1$ and all $k \geq 0$*

$$\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2(\Omega)} \leq d^{k+1} k^k.$$

Then $u \in B^1_\beta(\Omega)$.

Proof. The proof follows by interpreting the results from [21]. Following this reference, we define ε-neighborhoods at an edge e, at a vertex v, respectively at e and v:

$$\begin{aligned}\omega_v &= \{x \in \Omega : \mathrm{dist}(x,v) < \varepsilon \text{ and } \mathrm{dist}(x,\partial\Omega) \geq \varepsilon \mathrm{dist}(x,v)\},\\ \omega_{ve} &= \{x \in \Omega : \mathrm{dist}(x,v) < \varepsilon \text{ and } \mathrm{dist}(x,e) < \varepsilon \mathrm{dist}(x,v)\},\\ \omega_e &= \{x \in \Omega : \mathrm{dist}(x,e) < \varepsilon^2 \text{ and } \mathrm{dist}(x,e) \geq \varepsilon \quad \forall \text{ vertices } e\}.\end{aligned}$$

Estimates near edge: We consider the definitions of $H^{k,1}_\beta(\Omega)$ and $\Phi_{\beta,(\alpha_1,\alpha_2),1}$ in a strip ω_e at the x-axis. In ω_e the weight function $\Phi_{\beta,(\alpha_1,\alpha_2),1}$ is equivalent to the local weight function

$$\Phi^{\omega_e}_{\beta,(\alpha_1,\alpha_2),1}(x,y) = y^{\beta+\alpha_2-1}.$$

We observe by choosing $\beta = \frac{1}{2} - s + \varepsilon$, $\alpha_1 = p_\parallel$, $\alpha_2 = p_\perp$ that

$$\|y^{\beta+\alpha_2-1}\partial_x^{\alpha_1}\partial_y^{\alpha_2} u\|_{L^2(\omega_e)} = \|y^{p_\perp - \frac{1}{2} - s + \varepsilon}\partial_x^{p_\perp}\partial_y^{p_\parallel} u\|_{L^2(\omega_e)}.$$

Similarly, in an edge-vertex neighborhood ω_{ve} the weight function $\Phi_{\beta,(\alpha_1,\alpha_2),1}$ is equivalent to the local weight function

$$\Phi^{\omega_e}_{\beta,(\alpha_1,\alpha_2),1}(x,y) = x^{\alpha_1} y^{\beta+\alpha_2-1},$$

provided $\alpha_1, \alpha_2 \geq 1$. Now we observe with $\beta = \frac{1}{2} - s + \varepsilon$, $\alpha_1 = p_\parallel$, $\alpha_2 = p_\perp$ that

$$\|x^{\alpha_1} y^{\beta+\alpha_2-1}\partial_x^{\alpha_1}\partial_y^{\alpha_2} u\|_{L^2(\omega_{ve})} = \|x^{p_\parallel} y^{p_\perp - \frac{1}{2} - s + \varepsilon}\partial_x^{p_\perp}\partial_y^{p_\parallel} u\|_{L^2(\omega_{ve})}.$$

Hence the result from [23] implies that in a neighborhood $\tilde{Q}$ of the edges $u \in B^1_\beta(\tilde{Q})$ if and only if $u \in H^{k,1}_\beta(\tilde{Q})$ for all $k \geq 1$ and $\|\Phi_{\beta,\alpha,1} D^\alpha u\|_{L^2(\tilde{Q})} \leq C d^{k-1}(k-1)!$ for all $|\alpha| = k = 1, 2, \ldots$, where $C, d \geq 1$ are constants which are independent of k. This follows from

$$\|\Phi_{\beta,\alpha,1} D^\alpha u\|_{L^2(\omega_e)} + \|\Phi_{\beta,\alpha,1} D^\alpha u\|_{L^2(\omega_{ve})} \leq C d^{k-1}(k-1)!\ .$$

Estimates near corner: To describe the solution in the vertex neighborhood ω_v we use the weighted Sobolev space $\hat{H}^{k,1}_\beta(Q)$, which consists of all $u \in L^2(Q)$ such that $\|\tilde{\Phi}_{\beta,\alpha,1}\mathcal{D}^\alpha u\|_{L^2(Q)} < \infty$ for all $0 \leq |\alpha| \leq k$. Here $\mathcal{D}^\alpha = \partial_r^{\alpha_r}\partial_\theta^{\alpha_\theta}$ and

$$\tilde{\Phi}_{\beta,\alpha,1} = r^{\beta+\alpha_r-1}(\sin(\theta)\sin(\omega-\theta))^{(\beta+\alpha_\theta-1)_+},$$

with $(a)_+ = \max\{a, 0\}$. Define $\hat{Q} = [0, \omega]_\theta \times [0, R]_r$. In $\omega_v = [\theta_1, \theta_2]_\theta \times [0, R] \subset \tilde{Q}$ we define

$$\tilde{\Phi}_{\beta,(\alpha_r,\alpha_\theta),1} = r^{\beta+\alpha_r-1+\alpha_\theta}.$$

Again we take $\beta = \frac{1}{2} - s + \varepsilon$ and observe for $\alpha_r = 1$, $\alpha_\theta = 0$ with $\partial_r = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \cos(\theta)\partial_x + \sin(\theta)\partial_y$ that for $\alpha_1 + \alpha_2 = 1$

$$\begin{aligned}\|r^{\beta+\alpha_r-1}\partial_r^{\alpha_r}\partial_\theta^{\alpha_\theta}u\|_{L^2(\omega_v)} &= \|r^\beta\partial_r u\|_{L^2(\omega_v)} = \|r^\beta(\cos(\theta)\partial_x + \sin(\theta)\partial_y)u\|_{L^2(\omega_v)} \\ &\leq \|r^{\beta+\alpha_r-1}\partial_x^{\alpha_1}\partial_y^{\alpha_2}u\|_{L^2(\omega_v)} = \|r^{|\alpha|-\frac{1}{2}-s+\varepsilon}\partial_x^{\alpha_1}\partial_y^{\alpha_2}u\|_{L^2(\omega_v)}.\end{aligned}$$

Consider now $\alpha_r = 1$, $\alpha_\theta = 1$ and note that $\partial_\theta u = \frac{1}{\sin(\omega)}((x + y\cos(\omega))\partial_y - (x\cos(\omega) + y)\partial_x)u$. Then we obtain similarly

$$\begin{aligned}&\|r^{\beta+1}(\sin(\theta))^\beta\partial_r^{\alpha_r}\partial_\theta^{\alpha_\theta}u\|_{L^2(\omega_v)} \\ &= \|r^{\beta+1}(\cos(\theta)\partial_x + \sin(\theta)\partial_y)\frac{1}{\sin(\omega)}((x + y\cos(\omega))\partial_y - (x\cos(\omega) + y)\partial_x)u\|_{L^2(\omega_v)} \\ &\leq \|r^{\beta+1}\partial_x\partial_y u\|_{L^2(\omega_v)} = \|r^{|\alpha|-\frac{1}{2}-s+\varepsilon}\partial_x^{\alpha_1}\partial_y^{\alpha_2}u\|_{L^2(\omega_v)}.\end{aligned}$$

The result follows for all α_r, α_θ by induction.
The argument in ω_{ve} combines the arguments in ω_v and ω_e.
Now we observe from [38] that B^1_β in Cartesian coordinates is equivalent to B^1_β in polar coordinates. Hence Theorem 2.1 in [21] gives $u \in B^1_\beta(Q)$ - this is what we need for our approximation analysis. □

Corollary 4.6. *Let Ω be a polygonal domain, $u \in \widetilde{H}^s(\Omega)$ be a solution to* (1) *with right hand side $f \in C^\infty(\overline{\Omega})$ and $u_{hp} \in \widetilde{\mathbb{S}}_{h\mathbf{p}}$ be the finite element approximation on a geometrically graded mesh. Assume that for some $d \geq 1$ and all $k \geq 0$*

$$\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2(\Omega)} \leq d^{k+1}k^k.$$

Then

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \lesssim e^{-CN^{1/4}},$$

where C is independent of number of degrees of freedom N.

In the numerical experiments below we observe that for sufficiently large N the numerical error of the *hp* version on geometrically graded meshes behaves like $\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \lesssim e^{-CN^{1/4}}$, as predicted by Corollary 4.6.

5 Proof of approximation results on quasi-uniform meshes

We now discuss the proof of Theorem 4.1 from Subsection 4.1. We use the notation of Theorem 3.2.

Edge-vertex singularities. Let $e \in E$ be the edge of Ω with neighbouring vertices v, w. Let A_e be the union of all elements at the edge e. We denote by ℓ_v and ℓ_w the edges of ∂A_e such that $\bar{\ell}_v \cap \bar{e} = \{v\}$ and $\bar{\ell}_w \cap \bar{e} = \{w\}$.
Let us consider the cut-off functions χ^v and χ^{ev} which appear in the expressions for the edge-vertex singularities u_1^{ev} and u_2^{ev}. We take the supports of these cut-off functions as follows:

$$\begin{aligned}\operatorname{supp}\chi^v &\subset [0, 2\tau_v] \text{ with } 0 < \tau_v < \min\left\{\frac{1}{4}\operatorname{dist}\{v, w\}, \frac{1}{2}\right\} \\ \operatorname{supp}\chi^{ev} &\subset \left[0, \frac{3}{2}\beta_v\right] \text{ with } 0 < \beta_v \leq \min\left\{\frac{1}{2}\theta_0, \frac{1}{2}\omega_v, \frac{\pi}{8}\right\}\end{aligned} \tag{34}$$

where θ_0 is the minimal angle of the triangles τ_h in the mesh $\mathcal{T}_h$. Then u_1^{ev} and u_2^{ev} vanish outside the sector $S = \left\{(r_v, \theta_v)\,; 0 < r_v < 2\tau_v, 0 < \theta_v < \frac{3}{2}\beta_v\right\}$, in particular, $u_1^{ev} = u_2^{ev} = 0$ on $\ell_v \cup \ell_w$.

Lemma 5.1. *Let $u = u_1^{ev}$ be the edge-vertex singular function as in Theorem 3.2. Then there exists $u_{hp} \in \widetilde{\mathbb{V}}^{h,p}(\Omega)$ with $p \geq \lambda = \lambda_1^v$ such that*

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \leq Ch^{1/2}p^{-1}\left(1 + \log(p/h)\right)^{\beta+\nu},$$

where, with the notation from Theorem 3.2,

$$\beta = \begin{cases} q_1^v + s + \frac{1}{2} & \text{if } \lambda_1^v = \gamma_1^e - \frac{1}{2} \\ q_1^v + s & \text{otherwise} \end{cases}, \nu = \begin{cases} \frac{1}{2} & \text{if } p = s - \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

The proof is a straight-forward extension of Theorem 5.1 in [8] to Sobolev exponent s, using the singular functions from the expansion of the solution given in Theorem 3.2. Details can be found in [47].

Lemma 5.2. *Let $u = u_2^{ev}$ be the edge-vertex singular function as in Theorem 3.2. Then there exists $u_{hp} \in \widetilde{\mathbb{V}}^{h,p}(\Omega)$ with $p \geq s - \frac{1}{2}$ such that*

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \leq Ch^{1/2}p^{-1}\left(1 + \log(p/h)\right)^{\beta+\nu},$$

where $\beta = s_1^e \in \mathbb{N}_0$, $\nu = \frac{1}{2}$ if $p = s - \frac{1}{2}$ and $\nu = 0$ otherwise.

Again the proof is a straight-forward extension of Theorem 5.2 in [8] to Sobolev exponent s, using the singular functions from the expansion of the solution given in Theorem 3.2. Details can be found in [47].

Remark 5.3. Note that since the cut–off functions in u_2^{ev} can be rewritten by using (31), we can apply Lemma 5.2 to obtain estimates for the edge singular functions u^e.

Vertex singularities. Now we approximate the vertex singularities u^v as in Theorem 3.2. Let $v \in V$ be a vertex of Ω.

Lemma 5.4. *Let $u = u^v$ be the vertex singularity function as in Theorem 3.2. Then there exists $u_{hp} \in \widetilde{\mathbb{V}}^{h,p}(\Omega)$ with $p \geq \lambda$ such that*

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \leq Ch^{\lambda+1-s}p^{-2(\lambda+1-s)}(1 + \log(p/h))^{\beta+\nu}$$

where $\lambda = \lambda_1^v > 0$, $\beta = q_1^v \in \mathbb{N}_0$, $\nu = \frac{1}{2}$ if $p = \lambda$ and $\nu = 0$ otherwise.

The proof is a straight-forward extension of Theorem 6.1 in [8] to Sobolev exponent s, using the singular functions from the expansion of the solution given in Theorem 3.2. Details can be found in [47].

Edge singularities. As stated in Remark 5.3 estimates for the edge singularities u^e can be derived in the same way as for u_2^{ev}. Thus we only state the result for the edge singularities below.

Lemma 5.5. *Let $u = u^e$ be the edge singular function as in Theorem 3.2. Then there exists $u_{hp} \in \widetilde{\mathbb{V}}^{h,p}(\Omega)$ with $p \geq s - \frac{1}{2}$ such that*

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \leq Ch^{1/2}p^{-1}\left(1 + \log(p/h)\right)^{\beta+\nu},$$

where $\beta = s_1^e \in \mathbb{N}_0$, $\nu = \frac{1}{2}$ if $p = s - \frac{1}{2}$ and $\nu = 0$ otherwise.

The proof is a straight-forward extension of Theorem 6.2 in [8] to Sobolev exponent s, using the singular functions from the expansion of the solution given in Theorem 3.2. Details can be found in [47].

Regular part. We finally review results for the approximation of the regular remainder. For this it suffices to recall the approximation result for sufficiently smooth functions from [8, Proposition 4.1].

Lemma 5.6. *Let $u \in H^m(\Omega) \cap \widetilde{H}^1(\Omega)$ with $m > 1$. Then there exists $u_{hp} \in \widetilde{\mathbb{V}}^{h,p}(\Omega)$ such that for $s \in [0,1]$*

$$\|u - u_{hp}\|_{\widetilde{H}^s(\Omega)} \leq Ch^{\mu-s}p^{-(m-\tilde{s})}\|u\|_{H^m(\Gamma)}, \tag{35}$$

where $\mu = \min\{m, p+1\}$ and

$$\tilde{s} = \begin{cases} 1/2 & s \in [0,1/2) \\ 1/2+\varepsilon & s = 1/2 \\ s & s \in (1/2,1]. \end{cases} \tag{36}$$

6 Proof of approximation results on geometrically graded meshes

6.1 Piecewise polynomial approximation in weighted Sobolev spaces

We now discuss the proof of Theorem 4.4 in Subsection 4.2.

We first recall from [12] the spaces $H^t_\beta(Q)$, $\beta \in [0,1)$: $u \in H^t_0(Q)$ if $u \in L^2_{loc}(Q)$, for all $|\alpha| = [t]$

$$r^{|\alpha|-t}\partial^\alpha u \in L^2(Q),$$

and, if $\tau = t - [t] \in (0,1)$, for all $|\alpha| = [t]$ the function $\frac{\partial^\alpha u(x) - \partial^\alpha u(y)}{|x-y|^{\tau+1}}$ belongs to $L^2(Q \times Q)$.

We say that $u \in H^t_\beta(Q)$ if $r^\beta u \in H^t_0(Q)$, where r denotes the distance to the boundary. We will need the expression for the norm of $H^1_\beta(Q)$:

$$\|u\|^2_{H^1_\beta(Q)} = \|r^{\beta-1}u\|^2_{L^2(Q)} + \sum_{|\alpha|=1} \|\partial^\alpha(r^\beta u)\|^2_{L^2(Q)}.$$

Lemma 6.1. *Let $s \in (0,1)$. Then for all $u \in H^1_{1-s}(Q)$,*

$$\|u\|_{\widetilde{H}^s(Q)} \lesssim \|u\|_{H^1_{1-s}(Q)} \lesssim \|r^{-s}u\|_{L^2(Q)} + \sum_{|\alpha|=1} \|r^{1-s}\partial^\alpha u\|_{L^2(Q)}.$$

Proof. First note that $\widetilde{H}^s(Q) = H^s_0(Q)$ by Theorem AA.7 in [12], so that we need to show $\|u\|_{H^s_0(Q)} \lesssim \|u\|_{H^1_{1-s}(Q)}$. The inequality $\|u\|_{\widetilde{H}^s(Q)} \lesssim \|u\|_{H^1_{1-s}(Q)}$ now follows directly from the dyadic characterization of H^t_β-norms, Lemma AA.24 in [12], applied to $(t,\beta) = (s,0)$ and $(1, 1-s)$.

The remaining estimate, $\|u\|_{H^1_{1-s}(Q)} \lesssim \|r^{-s}u\|_{L^2(Q)} + \sum_{|\alpha|=1} \|r^{1-s}\partial^\alpha u\|_{L^2(Q)}$, follows from the expression for the norm of $H^1_\beta(Q)$, when $\beta = 1-s$, and the triangle inequality. □

Approximation near the edge:

Lemma 6.2. *Let $Q_1 = [a_1, b_1] \times [0, h]$, and $u \in H^{2,1}_\beta(Q_1)$. Define*

$$\bar{u}_h(x,y) = u(x,h) - \left(1 - \frac{y}{h}\right)\int_0^h \partial_z u(x,z)dz.$$

Then

$$\|y^{-s}(u(x,y)-\bar{u}_h(x,y))\|_{L^2(Q_1)} \lesssim h^{1-s-\beta}\|u\|_{H^{2,1}_\beta(Q_1)},$$
$$\|y^{-s}\partial_x(u(x,y)-\bar{u}_h(x,y))\|_{L^2(Q_1)} \lesssim h^{1-s-\beta}\|u\|_{H^{2,1}_\beta(Q_1)}.$$
$$\|y^{1-s}\partial_y(u(x,y)-\bar{u}_h(x,y))\|_{L^2(Q_1)} \lesssim h^{1-s-\beta}\|u\|_{H^{2,1}_\beta(Q_1)}.$$

Proof. For the first estimate, note that

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_0^h y^{-2s}(u(x,y)-\bar{u}_h(x,y))^2 dy\ dx\\
&=\int_{a_1}^{b_1}\int_0^h y^{-2s}\left(\int_y^h(\partial_z u(x,z)-\partial_z\bar{u}_h(x,z))dz\right)^2 dy\ dx\\
&\le \int_{a_1}^{b_1}\int_0^h y^{-2s}\int_y^h z^{-1}dz\int_0^h z(\partial_z u(x,z)-\partial_z\bar{u}_h(x,z))^2 dz\ dy\ dx\\
&\le \frac{h^{1-2s}}{(1-2s)^2}\int_{a_1}^{b_1}\int_0^h z\left(\partial_z u(x,z)-\frac{1}{h}\int_0^h\partial_w u(x,w)dw\right)^2 dz\ dx\\
&\le \frac{h^{1-2s}}{(1-2s)^2}\int_{a_1}^{b_1}\int_0^h z\left(\frac{1}{h}\int_0^h\int_w^z\partial_q^2 u(x,q)dq\ dw\right)^2 dz\ dx.
\end{aligned}$$

For a fixed $\epsilon>0$ sufficiently small, we observe

$$\begin{aligned}
&\frac{1}{h^2}\int_{a_1}^{b_1}\int_0^h z\int_0^h\left(\int_0^h w^{-\beta-\epsilon}w^{\beta+\epsilon}\int_w^z\partial_q^2 u(x,q)dq\ dw\right)^2 dz\ dx\\
\le\ &\frac{1}{h^2}\int_{a_1}^{b_1}\int_0^h z\frac{h^{1-2\beta-2\epsilon}}{1-2\beta-2\epsilon}\int_0^h w^{2\beta+2\epsilon}\left(\int_w^z\partial_q^2 u(x,q)dq\right)^2 dw\ dz\ dx,
\end{aligned}$$

yielding (with a constant C depending on ϵ)

$$\|y^{-s}(u(x,y)-\bar{u}_h(x,y))\|^2_{L^2(Q_1)} \le Ch^{1-2s+2-2\beta}\|u\|^2_{H^{2,1}_\beta(Q_1)}.$$

The proof of the second estimate in the Lemma is analogous, and we now prove the remaining, third estimate:

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_0^h y^{2-2s}(\partial_y u(x,y)-\partial_y\bar{u}_h(x,y))^2 dy\ dx\\
&=\int_{a_1}^{b_1}\int_0^h y^{2-2s}\left(\partial_y u(x,y)-\frac{1}{h}\int_0^h\partial_z u(x,y)dz\right)^2 dy\ dx\\
&=\int_{a_1}^{b_1}\int_0^h y^{2-2s}\left(\frac{1}{h}\int_0^h\int_z^y\partial_w^2 u(x,w)dw\ dz\right)^2 dy\ dx.
\end{aligned}$$

Next, we observe

$$\left(\int_0^h z^{-\beta-\epsilon}z^{\beta+\epsilon}\int_z^y\partial_w^2 u(x,w)dw\ dz\right)^2 \le \frac{h^{1-2\beta-2\epsilon}}{1-2\beta-2\epsilon}\int_0^h z^{2\beta+2\epsilon}\left(\int_z^y\partial_w^2 u(x,w)dw\right)^2 dz.$$

Furthermore

$$\begin{aligned}
&\int_0^h z^{2\beta+2\epsilon}\left(\int_z^y\partial_w^2 u(x,w)dw\right)^2 dz\\
&\le\left(\tfrac{y^{2\epsilon}}{2\epsilon(2\beta+2\epsilon+1)}+\tfrac{1}{1+2\beta}\left(y^{-2\beta-1}\tfrac{h^{2\beta+2\epsilon+1}}{2\beta+2\epsilon+1}-\tfrac{h^{2\epsilon}}{2\epsilon}\right)\right)\int_0^h\left(w^{1+\beta}\partial_w^2 u(x,w)\right)^2 dw.
\end{aligned}$$

Therefore we obtain

$$\int_{a_1}^{b_1}\int_0^h y^{2-2s}(\partial_y u(x,y)-\partial_y \bar{u}_h(x,y))^2 dy\ dx \le Ch^{2-2s-2\beta}\|u\|^2_{H^{2,1}_\beta(Q_1)}.$$

□

We further obtain near the corner the following two lemmas:

Lemma 6.3. *For $u \in H^{3,1}_\beta([0,h]^2)$ and $(x,y)\in[0,h]^2$ consider $I_y u(h,y) = u(h,h) - (1-\frac{y}{h})\int_0^h \partial_z u(h,z)dz$. Then there holds:*

$$\|y^{-s}\tfrac{x}{h}(u(h,y)-I_y u(h,y))\|_{L^2([0,h]^2)} \lesssim h^{1-s-\beta}\|u\|_{H^{3,1}_\beta([0,h]^2)},$$
$$\|y^{1-s}\partial_y\tfrac{x}{h}(u(h,y)-I_y u(h,y))\|_{L^2([0,h]^2)} \lesssim h^{1-s-\beta}\|u\|_{H^{3,1}_\beta([0,h]^2)},$$
$$\|x^{1-s}\partial_x\tfrac{x}{h}(u(h,y)-I_y u(h,y))\|_{L^2([0,h]^2)} \lesssim h^{1-s-\beta}\|u\|_{H^{3,1}_\beta([0,h]^2)}.$$

Proof. We only show the second estimate. The first and third estimate are shown in an analogous, simpler way.
Note that with the definition of $I_y u(h,y)$,

$$\begin{aligned}\|y^{1-s}\partial_y\tfrac{x}{h}(u(h,y)-I_y u(h,y))\|^2_{L^2([0,h]^2)} &= \int_0^h \tfrac{x^2}{h^2}\int_0^h y^{2-2s}\left(\partial_y(u(h,y)-I_y u(h,y))\right)^2 dy\ dx\\ &= \int_0^h \tfrac{x^2}{h^2}\int_0^h y^{2-2s}\left(\partial_y u(h,y)-\tfrac{1}{h}\int_0^h \partial_z u(h,z)dz\right)^2 dy\ dx.\end{aligned}$$

We use Lemma 7.7 in [38] to estimate the inner integral:

$$\int_0^h y^{2-2s}\left(\partial_y u(h,y)-\tfrac{1}{h}\int_0^h \partial_z u(h,z)dz\right)^2 dy \lesssim h^{2-2s-2\beta}\int_0^h \left(y^{1+\beta}\partial_y^2 u(h,y)\right)^2 dy.$$

It remains to adapt the proof of Lemma 7.11 in [38]:

$$\begin{aligned}&\|y^{1-s}\partial_y\tfrac{x}{h}(u(h,y)-I_y u(h,y))\|^2_{L^2([0,h]^2)}\\ &\lesssim h^{2-2s-2\beta}\int_0^h \tfrac{x^2}{h^2}\int_0^h \left(y^{1+\beta}\partial_y^2 u(h,y)\right)^2 dydx\\ &= h^{2-2s-2\beta}\int_0^h \tfrac{x^2}{h^2}\int_0^h y^{2+2\beta}\left(\int_x^h \partial_z\partial_y^2 u(z,y)dz+\partial_y^2 u(x,y)\right)^2 dy\ dx\\ &\lesssim h^{2-2s-2\beta}\int_0^h\int_0^h \left(\partial_y^2 u(x,y)\right)^2 dy\ dx\\ &\quad + h^{2-2s-2\beta}\int_0^h \tfrac{x^2}{h^2}\int_0^h y^{2+2\beta}\left(\int_x^h \partial_z\partial_y^2 u(z,y)dz\right)^2 dy\ dx\\ &\lesssim h^{2-2s-2\beta}\int_0^h\int_0^h \left(\partial_y^2 u(x,y)\right)^2 dy\ dx\\ &\quad + h^{2-2s-2\beta}\int_0^h \tfrac{x^2}{h^2}\int_x^h \tilde{z}^{-2}d\tilde{z}\ dx\int_0^h y^{2+2\beta}\int_0^h z^2\left(\partial_z\partial_y^2 u(z,y)\right)^2 dz\ dy.\end{aligned}$$

Due to $\int_0^h \frac{x^2}{h^2}\int_x^h \tilde{z}^{-2}d\tilde{z}\ dx = \frac{1}{6}$, we conclude the proof

$$\|y^{1-s}\partial_y\tfrac{x}{h}(u(h,y)-I_y u(h,y))\|^2_{L^2([0,h]^2)} \lesssim h^{2-2s-2\beta}\|u\|^2_{H^{3,1}_\beta([0,h]^2)}.$$

□

Approximation near the corner:

Lemma 6.4. *For $u \in H^{2,1}_\beta([0,h]^2)$ and $(x,y) \in [0,h]^2$, define*

$$\bar{u}^D(x,y) = u(x,y) + \frac{x}{h}\frac{y}{h}u(h,h) - \frac{x}{h}u(h,y) - \frac{y}{h}u(x,h)$$

and

$$\phi^D(x,y) = \frac{h-x}{h}\frac{h-y}{h}\phi_0.$$

Here, $\phi_0 = 0$ if $u|_{[0,1]\times\{0\}}$ or $u|_{\{0\}\times[0,1]} = 0$, and $\phi_0 = \frac{1}{h^2}\int_0^h\int_0^h u(x,y)dx\ dy$ otherwise. Then

$$\|y^{-s}(u^D(x,y) - \phi^D(x,y))\|_{L^2([0,h]^2)} \lesssim h^{1-s-\beta}\|u\|_{H^{2,1}_\beta([0,h]^2)},$$
$$\|y^{1-s}\partial_y(u^D(x,y) - \phi^D(x,y))\|_{L^2([0,h]^2)} \lesssim h^{1-s-\beta}\|u\|_{H^{2,1}_\beta([0,h]^2)},$$
$$\|x^{1-s}\partial_x(u^D(x,y) - \phi^D(x,y))\|_{L^2([0,h]^2)} \lesssim h^{1-s-\beta}\|u\|_{H^{2,1}_\beta([0,h]^2)}.$$

Proof. We estimate

$$\begin{aligned}\int_0^1\int_0^h y^{-2s}(u^D - \phi^D)^2 dy\ dx &= \int_0^h\int_0^h y^{-2s}(u^D(x,y) - \phi^D(x,y) - (u^D(h,y) - \phi^D(h,y)))^2 dy\ dx\\ &= \int_0^h\int_0^h y^{-2s}(-\int_x^h \partial_z(u^D(z,y) - \phi^D(z,y))dz)^2 dy\ dx\\ &\leq \int_0^h\int_0^h y^{-2s}\int_x^h z^{-2\beta}\int_x^h z^{2\beta}\left(\partial_z(u^D(z,y) - \phi^D(z,y))\right)^2 dz\ dy\ dx\\ &\leq Ch^{2-2s-2\beta}\|x^{2\beta}\partial_x(u^D - \phi^D)\|^2_{L^2([0,h]^2)}.\end{aligned}$$

For the remaining estimate, we observe

$$\begin{aligned}&\partial_x(u^D - \phi^D)\\ &= \frac{h-y}{h}\partial_x u(x,y) - \frac{1}{h}\frac{h-y}{h}\int_x^h \partial_z u(z,y)dz - \frac{y}{h}\int_y^h \partial_x\partial_z u(x,z)dz\\ &+ \frac{1}{h}\frac{y}{h}\int_x^h\int_y^h \partial_z\partial_w u(z,w)dw\ dz + \frac{1}{h}\frac{y}{h}\int_y^h \partial_z u(x,z)dz - -\frac{1}{h}\frac{h-y}{h}u(x,y) + \frac{1}{h}\frac{h-y}{h}\phi_0.\end{aligned}$$

We consider the following term. The others are estimated in a similar way.

$$\begin{aligned}&\int_0^h\int_0^h x^{2-2s}\left(\frac{1}{h}\frac{h-y}{h}\int_x^h \partial_z u(z,y)dz\right)^2 dy\ dx\\ &\leq \frac{1}{h^2}\int_0^h\int_0^h x^{2-2s}\int_x^h z^{-2\beta}dz\int_x^h z^{2\beta}(\partial_z u(x,z))^2 dz\ dy\ dx\\ &\leq Ch^{2-2s-2\beta}\int_0^h\int_0^h (z^\beta\partial_z u(x,z))^2 dz\ dy.\end{aligned}$$

This yields

$$\|x^{1-s}\partial_x(u^D - \phi^D)\|^2_{L^2([0,h]^2)} \leq Ch^{2-2s-2\beta}\|u\|^2_{H^{2,1}_\beta(Q_1)},$$

and therefore

$$\|u^D - \phi^D\|^2_{\tilde{H}^s([0,h]^2)} \leq Ch^{2-2s-2\beta}\|u\|^2_{H^{2,1}_\beta(Q_1)}.$$

□

Next we recall three results from [38].

Lemma 6.5. *Let $u \in H^{2,1}_\beta(Q)$, $Q = [0,1]^2$, $0 \le \beta < 1/2$. Then for all $(x_1, x_2), (y_1, y_2) \in Q$ there holds*

$$|u(x_1,x_2) - u(y_1,y_2)| \le \frac{C}{1-2\beta} \frac{|y_1^{1-2\beta} - x_1^{1-2\beta}| + |y_2^{1-2\beta} - x_2^{1-2\beta}|}{\min(x_1,y_1)^{1-\beta} + \min(x_2,y_2)^{1-\beta}} \|u\|_{H^{2,1}_\beta(Q)} \tag{37}$$

which implies $H^{2,1}_\beta(Q) \subset C^0(\bar{Q})$.

Theorem 6.6. *Let $Q_1 = (a_1,b_1) \times (0,h_2) \subseteq Q = [0,1]^2$ with $h_1 = b_1 - a_1 \le \lambda_1 a_1$, $\lambda_1 \ge 0$ and $a_1 > 0$. For $u \in H^{k+2,1}_\beta(Q)$ $(0 \le \beta < 1)$ with weight function $\Phi_{\beta,\alpha,1}(x,y)$ in (33) there exists a polynomial $\phi \in \mathcal{P}_{k_1,1}(Q_1)$ with $1 \le k_1 \le k$, such that for $0 \le |\alpha| = \alpha_1 + \alpha_2 \le 1$ there holds*

$$\begin{aligned} &\|D^\alpha(\bar{u}_{h_2} - \phi)\|^2_{L^2(Q_1)} \\ &\le C\, a_1^{-2\alpha_1}(a_1^{2(1-\beta)} + h_2^{2(1-\alpha_2-\beta)}) \frac{\Gamma(k_1 - s_1 + 1)}{\Gamma(k_1 + s_1 + 3 - 2\alpha_1)} \left(\frac{\lambda_1}{2}\right)^{2(s_1+1-\alpha_1)} |u|^2_{H^{s_1+2,1}_\beta(Q)} \end{aligned} \tag{38}$$

and for $z = 0$ or $z = h_2$

$$\begin{aligned} &\|\partial_x^{\alpha_1}(bu_{h_2}(x,z) - \phi(x,z))\|^2_{L^2(a_1,b_1)} \\ &\le C h_2^{-1} a_1^{-2\alpha_1} \left(h_2^{2(1-\beta)} + a_1^{2(1-\beta)}\right) \frac{\Gamma(k_1 - s_1 + 1)}{\Gamma(k_1 + s_1 + 3 - 2\alpha_1)} \left(\frac{\lambda_1}{2}\right)^{2(s_1+1-\alpha_1)} |u|^2_{H^{s_1+2,1}_\beta(Q)}. \end{aligned}$$

Here we take

$$\bar{u}_{h_2}(x,y) \quad = \quad u(x,b_2) - \left(1 - \frac{y}{h_2}\right) \int_{a_2}^{b_2} \partial_z u(x,z)\, dz \tag{39}$$

Here $s_1 \in \mathbb{R}$, arbitrary, $1 \le s_1 \le k_1$. and $H^{s_1+2,1}_\beta(Q)$ is the interpolation space $(H^{\tilde{k}_1+1,1}_\beta(Q), H^{\tilde{k}_1+2,1}_\beta(Q))_{\theta_1,\infty}$ for integers $\tilde{k}_1 = s_1 + 1 - \theta_1 \le k_1$, $0 \le \theta_1 \le 1$. The constant C is independent of k, but dependents on λ_1. Furthermore $\phi = u$ on the vertices of Q_1 and the tangential derivative of ϕ on the edges of Q_1 is the L^2-projection of the tangential derivative of u.

Theorem 6.7. *Let $Q_1 = (a_1,b_1) \times (a_2,b_2) \subset\subset Q$ with $h_1 = b_1 - a_1 \le \lambda_1 a_1$, $h_2 = b_2 - a_2 \le \lambda_2 a_2$, $\lambda_i \ge 0$, $(i = 1,2)$, and $u \in H^{k+3,1}_\beta(Q)$ with $\Phi_{\beta,\alpha,1}(x,y)$ as in (33). Then there exists a polynomial $\phi \in \mathcal{P}_{k_1,k_2}(Q_1)$ with $2 \le k_1, k_2 \le k$, such that for $0 \le \alpha_1, \alpha_2 \le 1$ there holds*

$$\|D^\alpha(u - \phi)\|^2_{L^2(Q_1)} \le C \sum_{i=1}^{2} a_i^{2(1-\alpha_i-\beta)} a_{3-i}^{-2\alpha_{3-i}} \frac{\Gamma(k_i - s_i + 1)}{\Gamma(k_i + s_i + 3 - 2\alpha_i)} \left(\frac{\lambda_i}{2}\right)^{2s_i} |u|^2_{H^{s_i+3,1}_\beta(Q)} \tag{40}$$

where $s_i \in \mathbb{R}$, $1 \le s_i \le k_i$ $(i = 1,2)$ and $H^{s_i+3,2}_\beta(Q)$ is the interpolation space $(H^{\tilde{k}_i+2,1}_\beta(Q), H^{\tilde{k}_i+3,1}_\beta(Q))_{\theta_i,\infty}$ for integers $\tilde{k}_i = s_i + 1 - \theta_i \le k_i$, $0 \le \theta_i \le 1$ and C is independent of k, but depends on λ_i $(i = 1,2)$. Furthermore $\phi = u$ on the vertices of Q_1.

6.2 Exponentially fast convergence: Proof of Theorem 4.4

Due to Lemma 6.5 we have $u \in C^0(\bar{Q})$, therefore the point evaluation of $u(x,y)$ is possible.

Let $0 \le \beta < 1/2$. Let I_y be the linear interpolation operator with respect to the variable y on the interval $[0,h_1]$. We split u into $u^A+u^B+u^C+u^D$ according to

$$u^A(x,y)=\begin{cases} u(x,y) & \text{if } (x,y)\in[h_1,1]^2,\\ I_y u(x,y) & \text{if } (x,y)\in[h_1,1]\times[0,h_1],\\ I_x u(x,y) & \text{if } (x,y)\in[0,h_1]\times[h_1,1],\\ \frac{x}{h_1}I_y u(h_1,y)+\frac{h_1-x}{h_1}\frac{y}{h_1}u(0,h_1) & \text{if } (x,y)\in[0,h_1]^2,\end{cases}$$

$$u^B(x,y)=\begin{cases} 0 & \text{if } (x,y)\in[0,1]\times[h_1,1],\\ u(x,y)-I_y u(x,y) & \text{if } (x,y)\in[h_1,1]\times[0,h_1],\\ \frac{x}{h_1}u(h_1,y)-\frac{x}{h_1}I_y u(h_1,y) & \text{if } (x,y)\in[0,h_1]^2,\end{cases}$$

$$u^C(x,y)=\begin{cases} 0 & \text{if } (x,y)\in[h_1,1]\times[0,1],\\ u(x,y)-I_x u(x,y) & \text{if } (x,y)\in[0,h_1]\times[h_1,1],\\ \frac{y}{h_1}u(x,h_1)-\frac{y}{h_1}I_x u(x,h_1) & \text{if } (x,y)\in[0,h_1]^2,\end{cases}$$

$$u^D(x,y)=\begin{cases} u(x,y)+\frac{x}{h_1}\frac{y}{h_1}u(h_1,h_1)-\frac{x}{h_1}u(h_1,y)-\frac{y}{h_1}u(x,h_1) & \text{if } (x,y)\in[0,h_1]^2,\\ 0 & \text{otherwise.}\end{cases}$$

We will construct the approximation function $\phi\in S^{p,1}(Q^n_\sigma)$ approximating u by constructing approximations $\phi^A,\phi^B,\phi^C,\phi^D\in S^{p,1}(Q^n_\sigma)$ which are approximating u^A,u^B,u^C,u^D such that $\phi=\phi^A+\phi^B+\phi^C+\phi^D$.

6.2.1 Construction of ϕ^A

First we note that u and u^A are identical on $[h_1,1]^2$, therefore also ϕ and ϕ^A will be identical on $[h_1,1]^2$. We define

$$\phi^A(x,y)=\frac{x}{h_1}\frac{y}{h_1}u(h_1,h_1)+\frac{x}{h_1}\frac{h_1-y}{h_1}u(h_1,0)+\frac{h_1-x}{h_1}\frac{y}{h_1}u(0,h_1)\qquad\text{on}\quad[0,h_1]^2\tag{41}$$

and we have $(u^A-\phi^A)|_{[0,h_1]^2}=0$.

Due to Theorem 6.6 we get polynomials $\psi_{k1}\in\mathcal{P}_{p_k1}(R_{k1})$ on the strip $\{(x,y):h_1\le x\le 1,0\le y\le h_1\}$ along the edge, which coincide at the corner points with u and which fulfill (for suitable s_k)

$$\begin{aligned}&\|D^\alpha(u^A-\psi_{k1})\|^2_{L^2(R_{k1})}\\ &\le C\,x_{k-1}^{-2\alpha_1}(x_{k-1}^{2(1-\beta)}+h_1^{2(1-\alpha_2-\beta)})\frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+3-2\alpha_1)}\left(\frac{\lambda}{2}\right)^{2(s_k+1-\alpha_1)}|u|^2_{H^{s_k+2,1}_\beta(Q)}.\end{aligned}\tag{42}$$

Due to Theorem 6.7 we get polynomials $\psi_{kl}\in\mathcal{P}_{p_kp_l}(R_{kl})$ with $1\le s_k\le p_k$ and $0\le\alpha_1,\alpha_2\le 1$ for the inner elements R_{kl}, $(2\le k,l\le n)$:

$$\begin{aligned}\|D^\alpha(u-\psi_{kl})\|^2_{L^2(R_{kl})}\le C\Bigg(&x_{k-1}^{2(1-\alpha_1-\beta)}x_{l-1}^{-2\alpha_2}\frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+3-2\alpha_1)}\left(\frac{\lambda}{2}\right)^{2s_k}|u|^2_{H^{s_k+3,1}_\beta(Q)}\\ &+x_{k-1}^{2(1-\alpha_1-\beta)}x_{l-1}^{-2\alpha_2}\frac{\Gamma(p_l-s_l+1)}{\Gamma(p_l+s_l+3-2\alpha_2)}\left(\frac{\lambda}{2}\right)^{2s_l}|u|^2_{H^{s_l+3,1}_\beta(Q)}\Bigg).\end{aligned}\tag{43}$$

For the construction of a continuous $\phi(x,y)\in S^p(Q^n_\sigma)$ we have to investigate the jumps of ψ_{kl} from R_{kl} to its neighboring elements. Let $1\le k\le n-1, 1\le l\le n$. Then $\psi_{k,l}$ and $\psi_{k+1,l}$ coincide with $u(x,y)$ at the points $(x,y)=(x_k,x_l)$ and (x_k,x_{l-1}). The difference $\tilde{w}^r_{kl}=(\psi_{k+1,l}-\psi_{k,l})|_{\gamma^r_{kl}}$ vanishes at the endpoints of the common side $\gamma^r_{kl}=\{x_k\}\times[x_{l-1},x_l]$ of R_{kl} and $R_{k+1,l}$. $\tilde{w}^r_{kl}$ is a polynomial of degree $\le p_l$ in y. Due to Lemma 7.5 in [38] there is a polynomial w_{kl}^r of degree p_l in y and of degree 1 in x such that for $x_{l-1} \le y \le x_l$ we have $w_{kl}^r(x_k, y) = \tilde{w}_{kl}^r(y)$. The polynomial w_{kl}^r vanishes at the other sides of R_{kl}. For $0 \le \alpha_1 + \alpha_2 \le 1$

$$\|D^\alpha w_{kl}^r(x,y)\|_{L^2(R_{kl})}^2 \tag{44}$$

$$\le C\,(h_k^{1-2\alpha_1} h_l^{2(1-2\alpha_2)} \|\partial_y \tilde{w}_{kl}^r(y)\|_{L^2(\gamma_{kl}^r)}^2$$

$$\le C\left(h_k^{1-2\alpha_1} h_l^{2(1-2\alpha_2)} (\|\partial_y(\psi_{kl} - u)\|_{L^2(\gamma_{kl}^r)}^2 + \|\partial_y(\psi_{k+1,l} - u)\|_{L^2(\gamma_{kl}^r)}^2)\right) \tag{45}$$

Analogously we have $\tilde{w}_{kl}^o = (\psi_{k+1,l} - \psi_{k,l})|_{\gamma_{kl}^o}$ at the common side $\gamma_{kl}^o = [x_{k-1}, x_k] \times \{x_l\}$ of R_{kl} and $R_{k,l+1}$.$\psi_{k,1}$ and $\psi_{k+1,1}(1 \le k \le n-1)$ coincide with $u(x,y)$ at the endpoints of the common side. By construction $\psi_{k,1}$ and $\psi_{k+1,1}$ are linear in y, i.e. there is no jump between $\psi_{k,1}$ and $\psi_{k+1,1}$. The same result holds for $\psi_{1,l}$ and $\psi_{1,l+1}(1 \le l \le n-1)$. Therefore we have $w_{k1}^r = 0$ for $1 \le k \le n-1$ and $w_{1l}^o = 0$ for $1 \le l \le n-1$.

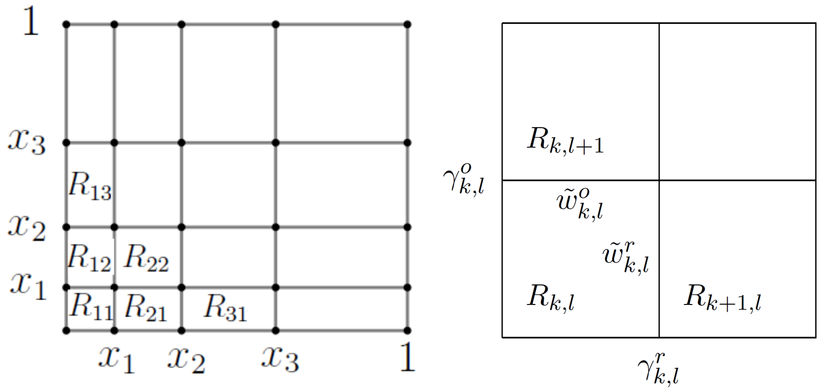


Figure 10: Mesh near corner (left) and element-to-element interfaces (right)

Let for $1 \le k, l \le n-1$

$$\phi_{k,l} = \psi_{k,l} + w_{kl}^r + w_{kl}^o. \tag{46}$$

Due to construction we have at the common side $\gamma_{kl}^r = x_k \times [x_{l-1}, x_l]$ of R_{kl} and $R_{k+1,l}$ for $1 \le k \le n-1, 1 \le l \le n$

$$(\phi_{k+1,l} - \phi_{k,l})|_{\gamma_{kl}^r} = (\psi_{k+1,l} - \psi_{k,l} + w_{k+1,l}^r - w_{kl}^r + w_{k+1l}^o - w_{kl}^o)|_{\gamma_{kl}^r} = (\psi_{k+1,l} - \psi_{k,l} - w_{k,l}^r)|_{\gamma_{kl}^r} = 0. \tag{47}$$

Analogously we have on the common side $\gamma_{kl}^o = [x_{k-1}, x_k] \times x_l$ of $R_{k,l+1}$ and $R_{k,l}$ for $1 \le k \le n, 1 \le l \le n-1$

$$(\phi_{k,l+1} - \phi_{k,l})|_{\gamma_{kl}^o} = (\psi_{k,l+1} - \psi_{k,l} + w_{k,l+1}^r - w_{kl}^r + w_{k,l+1}^o - w_{kl}^o)|_{\gamma_{kl}^o} = (\psi_{k,l+1} - \psi_{k,l} - w_{k,l}^o)|_{\gamma_{kl}^o} = 0. \tag{48}$$

Therefore there is a continuous function ϕ with $\phi|_{R_{kl}} = \phi_{k,l}$, i.e. $\phi \in S^{p,1}(Q_\sigma^n)$ and, using

Lemma 2.1,

$$
\begin{aligned}
\|D^\alpha(u^A-\phi^A)\|^2_{L^2(Q)} &= \sum_{k,l=1}^{n} \|D^\alpha(u^A-\psi_{kl}-w^r_{kl}-w^o_{kl})\|^2_{L^2(R_{kl})} \\
&\le 3\sum_{k,l=1}^{n} \|D^\alpha(u^A-\psi_{kl})\|^2_{L^2(R_{kl})} + 3\sum_{k=1}^{n-1}\sum_{l=2}^{n} \|D^\alpha w^r_{kl}\|^2_{L^2(R_{kl})} + 3\sum_{k=2}^{n}\sum_{l=1}^{n-1} \|D^\alpha w^o_{kl}\|^2_{L^2(R_{kl})} \\
&\le 3\sum_{k,l=1}^{n} \|D^\alpha(u^A-\psi_{kl})\|^2_{L^2(R_{kl})} \\
&\quad + C\sum_{k=1}^{n-1}\sum_{l=2}^{n} h_k^{1-2\alpha_1} h_l^{2(1-2\alpha_2)} (\|\partial_y(u^A-\psi_{kl})\|^2_{L^2(\gamma^o_{kl})} + \|\partial_y(u^A-\psi_{k+1,l})\|^2_{L^2(\gamma^o_{kl})}) \\
&\quad + C\sum_{k=2}^{n}\sum_{l=1}^{n-1} h_k^{2(1-\alpha_1)} h_l^{1-2\alpha_2} (\|\partial_x(u^A-\psi_{kl})\|^2_{L^2(\gamma^o_{kl})} + \|\partial_x(u^A-\psi_{k,l+1})\|^2_{L^2(\gamma^o_{kl})}). \qquad (49)
\end{aligned}
$$

In the following we estimate the terms on the side γ^o_{kl}. The terms on γ^r_{kl} can be estimated analogously. For $2 \le k,l \le n$ we have

$$
\begin{aligned}
& h_k^{2(1-\alpha_1)} h_l^{1-2\alpha_2} \|\partial_x(u^A-\psi_{kl})\|^2_{L^2(\gamma^o_{kl})} \\
& \le C h_k^{2(1-\alpha_1)} h_l^{1-2\alpha_2} (h_l^{-1}\|\partial_x(u^A-\psi_{kl})\|^2_{L^2(R_{kl})} + h_l \|\partial_x\partial_y(u^A-\psi_{kl})\|^2_{L^2(R_{kl})}) \\
& = C\left(h_k^{2(1-\alpha_1)} h_l^{-2\alpha_2}\|\partial_x(u^A-\psi_{kl})\|^2_{L^2(R_{kl})} + h_k^{2(1-\alpha_1)} h_l^{2(1-\alpha_2)}\|\partial_x\partial_y(u^A-\psi_{kl})\|^2_{L^2(R_{kl})}\right). \qquad (50)
\end{aligned}
$$

Therefore using Theorem 6.7 and $h_k = \lambda x_{k-1}$, we get

$$
\begin{aligned}
& h_k^{2(1-\alpha_1)} h_l^{1-2\alpha_2} \|\partial_x(u^A-\psi_{kl})\|^2_{L^2(\gamma^o_{kl})} \\
& \le C h_k^{2(1-\alpha_1)} h_l^{-2\alpha_2} \Bigg(x_{k-1}^{2(1-1-\beta)} x_{l-1}^{0} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
& \qquad + x_{l-1}^{2(1-0-\beta)} x_{k-1}^{-2} \frac{\Gamma(p_l-s_l+1)}{\Gamma(p_l+s_l+1)} \left(\frac{\lambda}{2}\right)^{2s_l} |u|^2_{H^{s_l+3,1}_\beta(Q)} \Bigg) \\
& \qquad + C h_k^{2(1-\alpha_1)} h_l^{1-2\alpha_2} \Bigg(x_{k-1}^{2(1-1-\beta)} x_{l-1}^{-2} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
& \qquad + x_{l-1}^{2(1-1-\beta)} x_{k-1}^{-2} \frac{\Gamma(p_l-s_l+1)}{\Gamma(p_l+s_l+1)} \left(\frac{\lambda}{2}\right)^{2s_l} |u|^2_{H^{s_l+3,1}_\beta(Q)} \Bigg) \\
& \le C \Bigg(x_{k-1}^{2(1-\alpha_1-\beta)} x_{l-1}^{-2\alpha_2} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
& \qquad + x_{k-1}^{-2\alpha_1} x_{l-1}^{2(1-\alpha_2-\beta)} \frac{\Gamma(p_l-s_l+1)}{\Gamma(p_l+s_l+1)} \left(\frac{\lambda}{2}\right)^{2s_l} |u|^2_{H^{s_l+3,1}_\beta(Q)} \Bigg).
\end{aligned}
$$

Hence,

$$\begin{aligned}
&\sum_{k=2}^{n}\sum_{l=2}^{n} h_k^{2(1-\alpha_1)} h_l^{1-2\alpha_2} \|\partial_x(u^A-\psi_{kl})\|^2_{L^2(\gamma^o_{kl})} \\
&\leq C\sum_{k=2}^{n}\sum_{l=2}^{n} x_{k-1}^{2(1-\alpha_1-\beta)} x_l^{-2\alpha_2} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
&\quad + C\sum_{k=2}^{n}\sum_{l=2}^{n} x_{k-1}^{-2\alpha_1} x_{l-1}^{2(1-\alpha_2-\beta)} \frac{\Gamma(p_l-s_l+1)}{\Gamma(p_l+s_l+1)} \left(\frac{\lambda}{2}\right)^{2s_l} |u|^2_{H^{s_l+3,1}_\beta(Q)}.
\end{aligned} \tag{51}$$

The terms in (51) are of the form (43). Due to the symmetry of (51) in k and l, resp. α_1 and α_2, it is sufficient to investigate the following term

$$\sum_{k=2}^{n}\sum_{l=2}^{n} x_{k-1}^{2(1-\alpha_1-\beta)} x_l^{-2\alpha_2} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)}$$

for $(\alpha_1,\alpha_2)=(0,0)$, $(1,0)$ and $(0,1)$, respectively.
When $\alpha_1=\alpha_2=0$, we use $x_k=\sigma^{n-k}$ to obtain

$$\begin{aligned}
&\sum_{k=2}^{n}\sum_{l=2}^{n} x_{k-1}^{2(1-\beta)} x_l^{0} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
&= (n-1)\sigma^{2(1-\beta)(n-1)} \sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)}.
\end{aligned}$$

When $\alpha_1=1$, $\alpha_2=0$, we similarly have

$$\begin{aligned}
&\sum_{k=2}^{n}\sum_{l=2}^{n} x_{k-1}^{-2\beta} x_l^{0} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
&= (n-1)\sigma^{-2\beta(n-1)} \sum_{k=2}^{n} \sigma^{-2\beta(-k+2)} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)}.
\end{aligned}$$

When $\alpha_1=0$, $\alpha_2=1$ we use $\sum_{l=2}^n x_l^{-2}=\sum_{l=2}^n \sigma^{-2(n-l+1)} \lesssim \sigma^{-2(n-1)}$ and obtain

$$\begin{aligned}
&\sum_{k=2}^{n}\sum_{l=2}^{n} x_{k-1}^{2(1-\beta)} x_l^{-2} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
&= \left(\sum_{l=2}^{n} x_l^{-2}\right) \sum_{k=2}^{n} x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
&\leq (n-1)\sigma^{-2(n-1)}\sigma^{2(1-\beta)(n-1)} \sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)} \\
&= (n-1)\sigma^{-2\beta(n-1)} \sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} \frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,1}_\beta(Q)}.
\end{aligned}$$

Now we are investigating the terms $\|\partial_x(u^A-\psi_{kl})\|^2_{L^2(\gamma^o_{kl})}$ for the strips at the edges in (49) separately. Using Theorem 6.6, $h_1 = x_1$ and $|u|_{H_\beta^{s_k+2,1}(Q)} \leq |u|_{H_\beta^{s_k+3,1}(Q)}$, we have

$$\begin{aligned}
&\sum_{k=2}^{n} h_k^{2(1-\alpha_1)} h_1^{1-2\alpha_2} \|\partial_x(u^A - \psi_{kl})\|^2_{L^2(\gamma^o_{kl})} \\
&\leq C \sum_{k=2}^{n} h_k^{2(1-\alpha_1)} h_1^{1-2\alpha_2} h_1^{-1} x_{k-1}^{-2} \left(h_1^{2(1-\beta)} + x_{k-1}^{2(1-\beta)}\right) \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H_\beta^{s_k+2,1}(Q)} \\
&\leq C \sum_{k=2}^{n} x_{k-1}^{-2\alpha_1} x_1^{2(1-\beta-\alpha_2)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H_\beta^{s_k+2,1}(Q)} \\
&\quad + C \sum_{k=2}^{n} x_{k-1}^{2(1-\beta-\alpha_1)} x_1^{-2\alpha_2} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H_\beta^{s_k+3,1}(Q)}. \qquad (52)
\end{aligned}$$

The terms in (52) are of the form (43), and they are the leading terms in the summation of (51). Therefore the bounds obtained in (51) equally apply to (52) and (43).

Due to $u \in B^1_\beta(Q)$ we have now $u \in H_\beta^{s_k+3,1}(Q)$ with $|u|_{H_\beta^{s_k+3,1}(Q)} \leq C d^{s_k+2}\Gamma(s_k+3)$. Therefore we obtain with $\varrho = \max(1, \lambda)$

$$\begin{aligned}
&\sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H_\beta^{s_k+3,1}(Q)} \\
&\leq \sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\varrho d}{2}\right)^{2(s_k+2)} \Gamma(s_k+3)^2. \qquad (53)
\end{aligned}$$

On the other hand setting

$$F(\alpha, d) = \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}} \left(\frac{\alpha d}{2}\right)^{2\alpha} \qquad (54)$$

there holds for $s_k = \alpha p_k$ (see [32])

$$\frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\varrho d}{2}\right)^{2s_k+2} (\Gamma(s_k+3))^2 \leq C\, F(\alpha, \varrho d)^{p_k}\, p_k^{2-(-2)+0\cdot 2+1} = C\, F(\alpha, \varrho d)^{p_k}\, p_k^5$$

where $p_k = max(2, 1 + [\mu(k-1)])$ for $\mu > 0$ $(k = 2, \ldots, n)$. Setting $\alpha_k = \max(1/p_k, \alpha_{\min})$, $k = 2, \ldots, n$, with $\alpha_{\min} = \frac{2}{\sqrt{4+\varrho^2 d^2}}$, we get that (53) is bounded by

$$\sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} F(\alpha_k, \varrho d)^{p_k}\, p_k^5. \qquad (55)$$

Let $F_{\min} := F(\alpha_{\min}, \varrho d)$ and

$$\mu > \frac{2(1-\beta)\log\sigma}{\log F_{\min}} \qquad (56)$$

and let k_0 be defined by the equation $p_{k_0} = \left[\frac{1}{\alpha_{\min}}\right] + 1$. Then k_0 is bounded, yielding

$$p_{k_0} = [\mu(k_0 - 1)] \leq \frac{1}{\alpha_{\min}} + 2. \qquad (57)$$

Therefore we can bound (55) by

$$\sum_{k=2}^{k_0} \sigma^{2(1-\beta)(-k+2)} F(1/p_k, \varrho d)^{p_k}\, p_k^5 + \sum_{k=k_0+1}^{n} \sigma^{2(1-\beta)(-k+2)} (F_{\min})^{p_k} p_k^5. \qquad (58)$$

There holds

$$\sigma^{2(1-\beta)(1-k)} F_{\min}^{p_k} = \sigma^{2(1-\beta)} \frac{F_{\min}^{[\mu(k-1)]+1}}{\sigma^{2(1-\beta)k}} \leq C\, \sigma^{2(1-\beta)} \left(\frac{F_{\min}^{\mu}}{\sigma^{2(1-\beta)}} \right)^k \tag{59}$$

and $F_{\min}^{\mu} < \sigma^{2(1-\beta)}$, due to $F_{\min} < 1$ (see (56) and Theorem 5.1 in [32].) Therefore we have $q := \frac{F_{\min}^{\mu}}{\sigma^{2(1-\beta)}} < 1$ and $\sum_{k>k_0} q^k k^5 < \infty$. Hence the series on the right hand side in (58) is bounded. Altogether this yields the estimates

$$\|u^A - \phi^A\|_{L^2(Q)}^2 \leq C\,(n-1)^{1/2} \sigma^{(1-\beta)(n-1)} \tag{60}$$

$$\|u^A - \phi^A\|_{H^1(Q)}^2 \leq C\,(n-1)^{1/2} \sigma^{-\beta(n-1)} \tag{61}$$

Thus we have by interpolation

$$\|u^A - \phi^A\|_{H^s(Q)}^2 \leq C\, e^{-2bn} \tag{62}$$

for $n \geq n_0$ with a fixed integer n_0 and with $b = -((\log n_0)/n_0 + (2(1-s)-\beta)\log \sigma) > 0$. For the number of degrees of freedom $N = \dim S^p(Q_\sigma^n)$ we have

$$N \;=\; \left(2 + \sum_{i=1}^{n-1} (1 + max(2, 1 + [\mu i]))\right)^2 \leq \left(2(n+1) + \mu \frac{n(n-1)}{2}\right)^2 \leq C\, n^4.$$

Finally we get

$$\|u^A - \phi^A\|_{\widetilde{H}^s(Q)} \simeq \|u^A - \phi^A\|_{H^s(Q)} \leq C\, e^{-bN^{1/4}}. \tag{63}$$

6.2.2 Construction of ϕ^B and ϕ^C

Note that $u = u^A + u^B$ on $[h_1, 1] \times [0, h_1]$, because $u^C + u^D = 0$ there. Thus $u - \phi = u^A + u^B - (\phi^A + \phi^B) = u^A - \phi^A + u^B)$, choosing $\phi^B = 0$. Hence,

$$\|u - \phi\|_{H^s([h_1,1]\times[0,h_1])} \lesssim \|u^A - \phi^A\|_{H^s([h_1,1]\times[0,h_1])} + \|u^B\|_{H^s([h_1,1]\times[0,h_1])}.$$

Note that the first term on the right hand side was already treated above. For the second term we know $u^B = u(x,y) - I_y u(x,y)$ on $[h_1, 1] \times [0, h_1]$. Therefore, Lemma 6.2 gives together with Lemma 6.1 that

$$\|u^B\|_{H^s([h_1,1]\times[0,h_1])} \lesssim h_1^{\frac{3}{2}-s-\beta} \|u\|_{H_\beta^{2,1}([h_1,1]\times[0,h_1])}\ .$$

Furthermore, for $u^B(x,y) = \frac{x}{h_1} u(h_1, y) - \frac{x}{h_1} I_h u(h_1, y)$ in $[0, h_1]^2$ we find

$$\|y^{1-s} \partial_y u^B\|_{L^2(Q)} = \|y^{1-s} \partial_y u^B\|_{L^2([0,1]\times[0,h_1])} \lesssim h^{1-s-\beta} \|u\|_{H_\beta^{3,1}(Q)}.$$

Using (for simplicity, $s \neq \frac{1}{2}$) the equivalence of $\|u - \phi\|_{\widetilde{H}^s}$ and $\|u - \phi\|_{H^s}$, Lemma 2.1 gives,

$$\|u^B\|_{\widetilde{H}^s(Q)} \lesssim h_1^{1-s-\beta} \|u\|_{H_\beta^{3,1}([h_1,1]\times[0,h_1])}\ .$$

As $h_1 = \sigma^{n-1}$, convergence again is exponential.

The corresponding result for u^C follows by symmetry.

6.2.3 Construction of ϕ^D

Note that on $[0, h_1]^2$ we have $u^A - \phi^A = 0$ and $u^B - \phi^B = 0$ with $\phi^B = 0$. Hence, $u - \phi = u^D - \phi^D$ on $[0, h_1]^2$, and Lemma 6.4 together with Lemma 6.1 gives

$$\|u - \phi\|_{H^s([0,h_1]^2)} = \|u^D - \phi^D\|_{H^s([0,h_1]^2)} \lesssim h_1^{1-s-\beta} \|u^B\|_{H^{2,1}_\beta([0,h_1]^2)},$$

and using (for simplicity, $s \neq \frac{1}{2}$) the equivalence of $\|u - \phi\|_{\widetilde{H}^s}$ and $\|u - \phi\|_{H^s}$ and Lemma 2.1,

$$\|u - \phi\|_{\widetilde{H}^s(Q)} \lesssim h_1^{1-s-\beta} \|u^B\|_{H^{2,1}_\beta([0,h_1]^2)}.$$

As $h_1 = \sigma^{n-1}$, convergence again is exponential.

Finally, note that on $[h_1, 1]^2$ we have $u = u^A$ and $\phi = \phi^A$, while $u^B = u^C = u^D = 0$. Hence $u - \phi = u^A - \phi^A$ on $[h_1, 1]^2$.

Combining all the above approximation results, we conclude

$$\|u - \phi\|_{\widetilde{H}^s(Q)} \lesssim \|u^A - \phi^A\|_{\widetilde{H}^s(Q)} + \|u^B\|_{\widetilde{H}^s(Q)} + \|u^C\|_{\widetilde{H}^s(Q)} + \|u^D - \phi^D\|_{\widetilde{H}^s(Q)} \lesssim Ce^{-bN^{1/4}}.$$

This concludes the proof of Theorem 4.4.

7 Implementation

We briefly address the details of the implementation for the bilinear form a from (4) for the *hp* version on quadrilateral elements. The implementation of the h version on quasi-uniform meshes is discussed in [28, 29].

Let $L_n(x)$ be Legendre polynomials over $[0, 1]$ with $L_0(x) = 1,\ L_1(x) = 2x - 1$ and for $n \geq 2$

$$L_n(x) = \frac{2n-1}{n}(2x - 1)L_{n-1}(x) - \frac{n-1}{n} L_{n-2}(x).$$

Further define integrated Legendre polynomials $\widetilde{L}_n(x)$ over $[0, 1]$ by $\widetilde{L}_0(x) = 1 - x,\ \widetilde{L}_1(x) = x$ and for $n \geq 2$

$$\widetilde{L}_n(x) = \frac{1}{2n-1} \left(L_n(x) - L_{n-2}(x) \right)$$

Note that for $n \geq 2$ the integrated Legendre polynomials $\widetilde{L}_n(x)$ vanish at the endpoints of $[0, 1]$.

Let $\mathbb{S}^p$ be the space of polynomials of degree at most p.

On the reference rectangular element $Q = [0, 1]^2$ we define the basis functions as tensor products of the integrated Legendre polynomials $\widetilde{L}_n(x)$,

$$\varphi_{k,l}|_Q(x) = \widetilde{L}_k(x_1)\widetilde{L}_l(x_2), \quad 0 \leq k \leq p_{x_1},\ 0 \leq l \leq p_{x_2},$$

and $\varphi_{k,l}(x) = 0$ outside Q, see for example [49]. Here $p_{x_1},\ p_{x_2}$ are the maximal polynomial degrees in x_1 and x_2 directions, respectively.

Let $\mathcal{Q}_h$ be a mesh of M rectangular elements τ_m and R boundary edges $e_r \in \partial\mathcal{Q}_h$. Let $\chi_m : Q \to \tau_m$ be an affine transformation from a reference element $Q = [0, 1]^2$ to the element $\tau_m \in \mathcal{Q}_h$ and $\chi_r : e \to e_r$ be an affine transformation from a reference edge $e = [0, 1]$ to the boundary edge $e_r \in \partial\mathcal{Q}_h$. Furthermore, let $\tilde{\varphi}^m_{k,l}|_{\tau_m}(x) := \varphi_{k,l}(\chi_m^{-1}(x))$ and let $\mathbf{p} = (p_1,\ p_2,\ \ldots,\ p_M)$ be a vector of polynomial pairs $p_m = (p_{m,x_1},\ p_{m,x_2})$ associated with

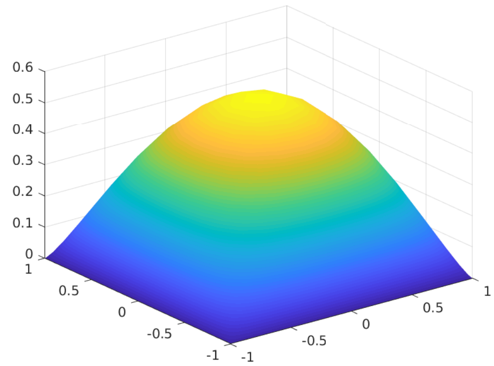


Figure 11: Solution for $s = 0.75$ and $p = 1$ in the square domain in Example 8.1.

an element τ_m for all elements in $\mathcal{Q}_h$.

We then implement the bilinear form a associated with the Dirichlet problem for the integral fractional Laplacian in $\widetilde{\mathbb{S}}_{h\mathbf{p}} \subset \widetilde{H}^s(\Omega)$, the finite element subspace given by

$$\widetilde{\mathbb{S}}_{h\mathbf{p}} = \{u \in \widetilde{H}^s(\Omega) : u \text{ continuous}, \ u|_{\tau_m} \in \mathcal{P}^{p_{m,x_1}, p_{m,x_2}}, \ \forall \tau_m \in \mathcal{T}_h\}.$$

Global basis functions are obtained by combining the local basis functions which do not vanish at the boundaries of the elements.

Entries of the stiffness matrix K are then obtained by combining the entries

$$\begin{aligned} K^{m,n}_{(i,j)(k,l)} =& \frac{c_{2,s}}{2} \iint_{\Omega\times\Omega} \frac{(\tilde{\varphi}^m_{i,j}(x) - \tilde{\varphi}^m_{i,j}(y))(\tilde{\varphi}^n_{k,l}(x) - \tilde{\varphi}^n_{k,l}(y))}{|x-y|^{2+2s}} \,\mathrm{dy}\,\mathrm{dx} \\ &+ \frac{c_{2,s}}{2s} \iint_{\Omega\times\partial\Omega} \frac{\tilde{\varphi}^m_{i,j}(x)\tilde{\varphi}^n_{k,l}(x)(x-y)\cdot n_y}{|x-y|^{2+2s}} \,\mathrm{dy}\,\mathrm{dx}\,. \end{aligned}$$

The integrals are computed using a composite graded quadrature as discussed in [11]. The computations are carried out on the quasi-uniform and geometrically graded meshes discussed in Section 4. Examples of geometrically graded meshes are depicted in Figures 7 and 8.

8 Numerical experiments

In this section all errors are measured in the norm of $\widetilde{H}^s(\Omega)$, i.e. the energy norm.

8.1 hp version on quasi-uniform meshes

Example 8.1. We consider the discretization of the Dirichlet problem (1) with $f = 1$ in the square domain $\Omega = [-1, 1]^2 \subset \mathbb{R}^2$ depicted in Figure 6 using quasi-uniform meshes. We examine fractional exponents $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{9}{10}$. A numerical solution for $s = \frac{3}{4}$ and $p = 1$ is shown in Figure 11.

The theoretically predicted convergence rates are confirmed in Figures 12 and 13. Figure 12 examines h–convergence in the energy norm for $s = \frac{3}{4}$ for different values of polynomial degree $p = 1, 2, 3$ on quasi–uniform meshes. The observed rates of convergence are approximately 0.5, in agreement with the theoretical approximation results, which predict an approximation error bounded by $h^{1/2}$.

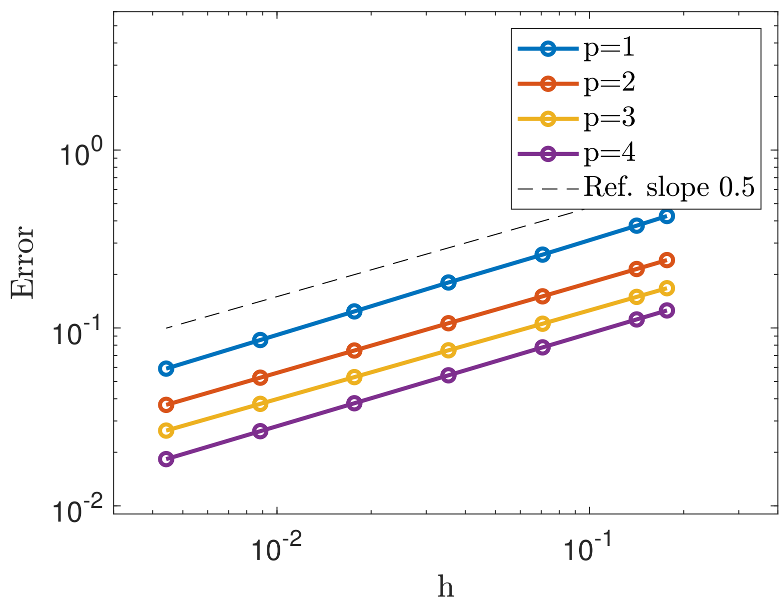


Figure 12: Convergence in h for $s = 0.75$ and different values of p on quasi–uniform meshes for Example 8.1.

Next, the convergence with respect to the polynomial degree p is examined in Figure 13 for different values of $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{9}{10}$ on a fixed quasi–uniform mesh with $h_{min} = 0.14$. Polynomial degrees up to $p = 9$ are considered. The observed convergence rates in p are close to 1: -1.01 ($s = \frac{1}{4}$), -0.996 ($s = \frac{1}{2}$), -1.00 ($s = \frac{3}{4}$), -1.01 ($s = \frac{9}{10}$). They agree with the theoretical approximation results, which predict an approximation error proportional to p^{-1}.

Example 8.2. We consider the discretization of the Dirichlet problem (1) with $f_1 = 1$, respectively $f_2 = \sin(2 + 0.2 \cdot (x - y))$, in the L-shaped domain $\Omega = [-1, 3]^2 \setminus [1, 3]^2 \subset \mathbb{R}^2$ depicted in Figure 7 using quasi-uniform meshes. We examine fractional exponents $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{9}{10}$. A numerical solution on a mesh with 3968 elements for $s = 0.75$ and $p = 1$ is shown in Figure 14 for f_1, respectively in Figure 15 for f_2.

For f_1, the theoretically predicted convergence rates are illustrated in Figures 16 and 18. Figure 16 examines h–convergence in the energy norm for $s = \frac{3}{4}$ for different values of polynomial degree $p = 1, 2, 3$ on quasi–uniform meshes. The observed rate of convergence is approximately 0.5. Like in Example 8.1 the rates agree with the theoretical expectations. Figure 17 shows the results of analogous experiments for the right hand side f_2, with identical conclusions as for the right hand side f_1.
The convergence with respect to the polynomial degree p is examined in Figure 18 for different values of $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{9}{10}$ for a fixed quasi–uniform mesh with $h = 0.16$. Polynomial degrees up to $p = 9$ are considered. The observed convergence rates in p are again compatible with -1.0: -1.012 ($s = \frac{1}{4}$), -1.014 ($s = \frac{1}{2}$), -1.004 ($s = \frac{3}{4}$), -1.017 ($s = \frac{9}{10}$). They agree with the theoretically expected convergence proportional to p^{-1}. Figure 19 shows the results of analogous experiments for the right hand side f_2, with identical conclusions as for the right hand side f_1.

8.2 *hp* version on geometrically graded meshes

Example 8.3. We consider the discretization of the Dirichlet problem (1) with $f = 1$ in the square domain $\Omega = [-1, 1]^2 \subset \mathbb{R}^2$ depicted in Figure 7 using rectangular geometrically graded

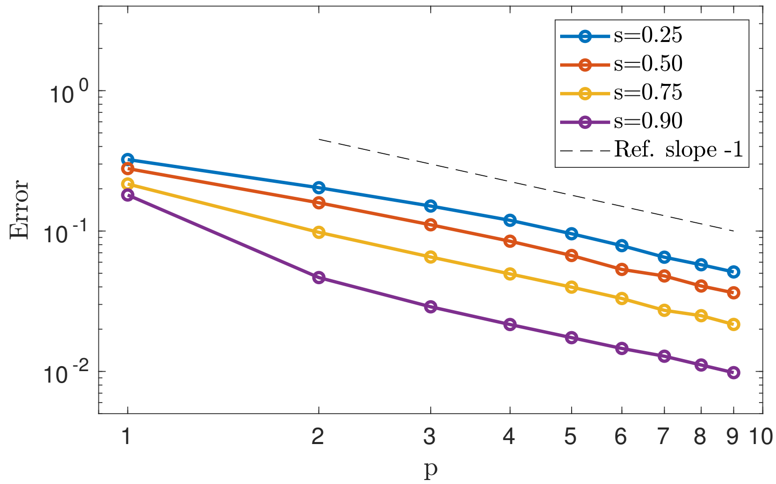


Figure 13: Convergence in p for different values of s on quasi–uniform meshes in Example 8.1.

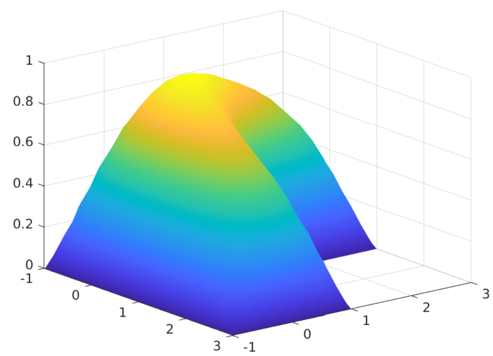


Figure 14: Solution for f_1, $s = 0.75$ in the L–shaped domain in Example 8.2.

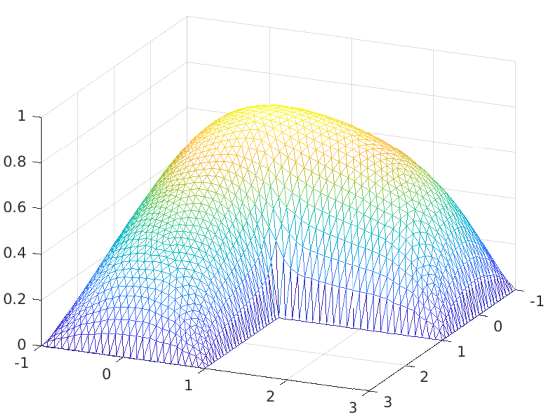


Figure 15: Solution for f_2, $s = 0.75$ in the L–shaped domain in Example 8.2.

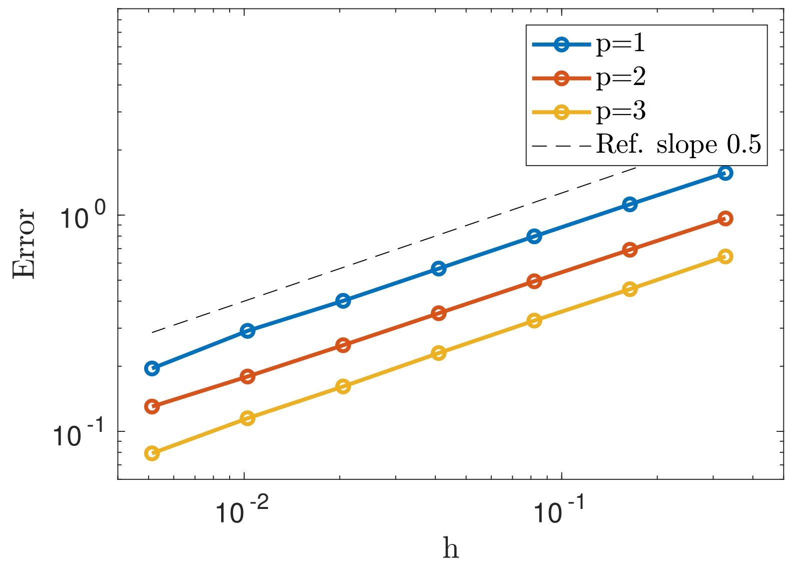


Figure 16: Convergence in h for f_1, $s = 0.75$ and different values of p on quasi–uniform meshes in Example 8.2.

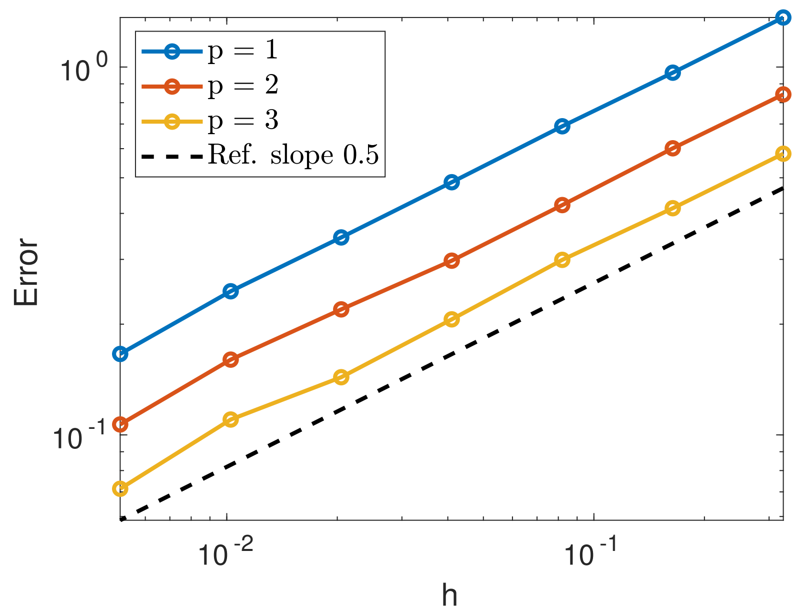


Figure 17: Convergence in h for f_2, $s = 0.75$ and different values of p on quasi–uniform meshes in Example 8.2.

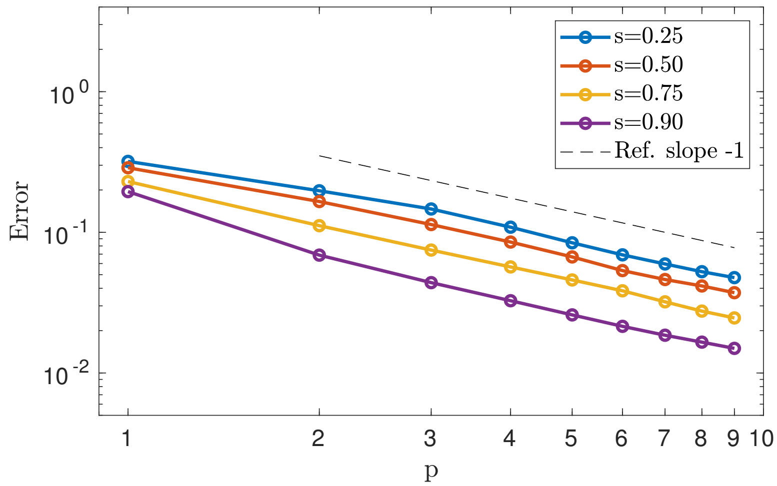


Figure 18: Convergence in p for f_1 and different values of s on quasi–uniform meshes in Example 8.2.

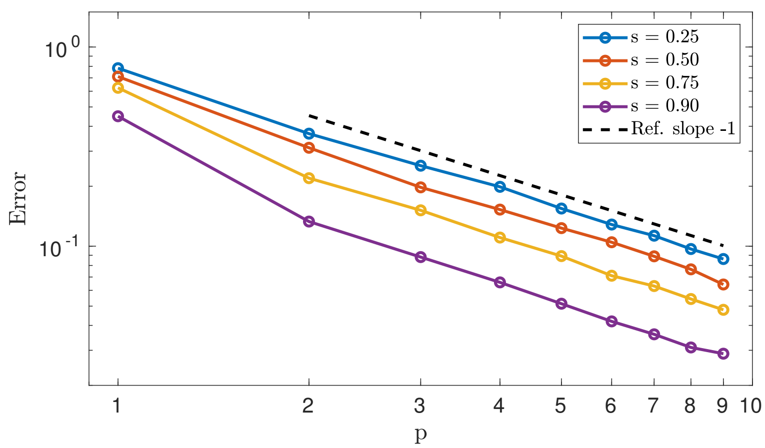
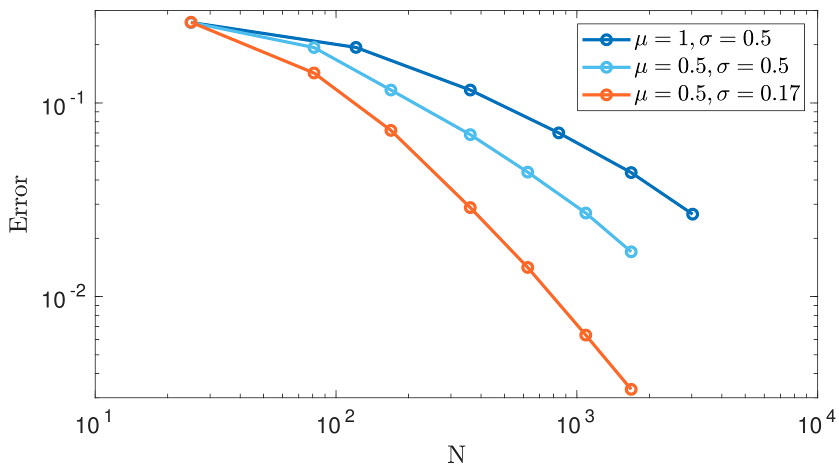


Figure 19: Convergence in p for f_2 and different values of s on quasi–uniform meshes in Example 8.2.

meshes with $\mu = 1,\ 0.5$ and $\sigma = 0.5,\ 0.17$. We examine fractional exponents $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. The number of degrees of freedom is denoted by N.

Figure 20 depicts the energy error of the hp version on a geometrically graded mesh with increasing N for the classical case $s = \frac{1}{2}$ in Example 8.3. The first plot suggests faster than algebraic decay by plotting the error as a function of N in logarithmic scales. The second plot shows that the error as a function of $N^{1/4}$ asymptotically follows a straight line in a semi-logarithmic scale for all the considered choices of μ and σ. This suggests that the expected size of the error is $\exp(-CN^{1/4})$ in all cases.

Figure 21 shows analogous results for $s = \frac{1}{4}, \frac{3}{4}$ in the semi-logarithmic scale also for higher values of N. As for $s = \frac{1}{2}$ the results asymptotically follow a straight line, suggesting that the expected size of the error is $\exp(-CN^{1/4})$ in all cases. Note that for all values of s the mesh grading parameter $\sigma = 0.17$ leads to the smallest errors.



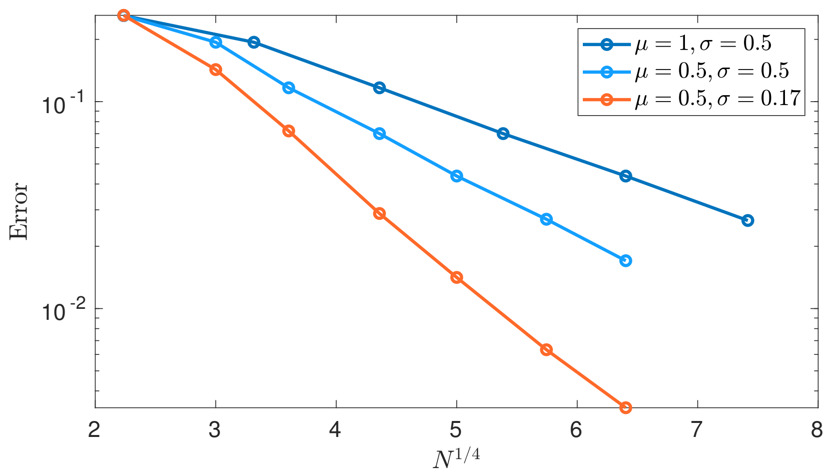


Figure 20: Convergence of hp approximations to u with respect to N and $N^{1/4}$ for problem (1) for $s = 0.5$, Example 8.3.

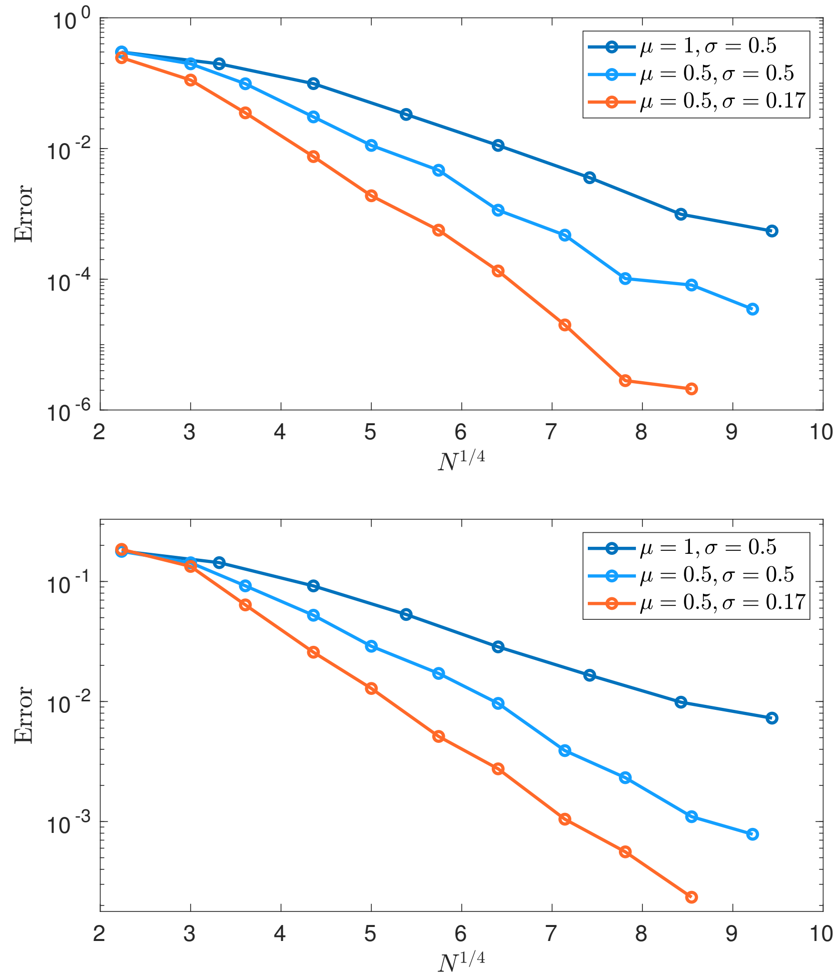


Figure 21: Convergence of hp approximations to u for Problem (1) for $s = 0.25,\ 0.75$, Example 8.3.

Example 8.4. We consider the discretization of the Dirichlet problem (1) with $f_1 = 1$, respectively $f_2 = \sin(2+0.2\cdot(x-y))$, in the L-shaped domain $\Omega = [-1,3]^2 \setminus [1,3]^2 \subset \mathbb{R}^2$ depicted in Figure 8 using geometrically graded meshes with $\mu = 1,\ 0.5$ and $\sigma = 0.5,\ 0.17$. We examine fractional exponents $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. The number of degrees of freedom is denoted by N.

For $f_1 = 1$, Figure 22 depicts the energy error of the hp version on a geometrically graded mesh with increasing N in Example 8.4 for different values of s. Like in Example 8.3, the error asymptotically follows straight lines in the semi-logarithmic plots for all the considered choices of μ and σ. It is therefore of the expected size $\exp(-CN^{1/4})$. For all values of s the mesh grading parameter $\sigma = 0.17$ again leads to the smallest errors.

Analogous results are obtained for $f_2 = \sin(2+0.2\cdot(x-y))$, as depicted in Figure 23.

9 Conclusions

In this work we initiate the study of p and hp versions of the finite element method for the integral fractional Laplacian in polygonal domains, combining theoretical analysis with extensive numerical experiments. Both quasi-uniform and geometrically graded discretizations are considered.

On quasi-uniform meshes the asymptotic expansion for the solution obtained in [28] near edges and corners allows us to obtain quasi-optimal estimates for the Galerkin error. In particular, the energy error of the hp version is $O(h^{1/2}p^{-1})$, and therefore the convergence in the polynomial degree p is twice as fast as the convergence in the mesh size h.

On a class of geometrically graded, quadrilateral meshes we prove exponentially fast convergence with respect to the number of degrees of freedom, by combining an analytic regularity result for the solution from [21] with ideas for the approximation in countably normed spaces based on [38].

The numerical results confirm the theoretically predicted convergence rates on quasi-uniform and on geometrically graded meshes. They illustrate the performance of the respective methods.

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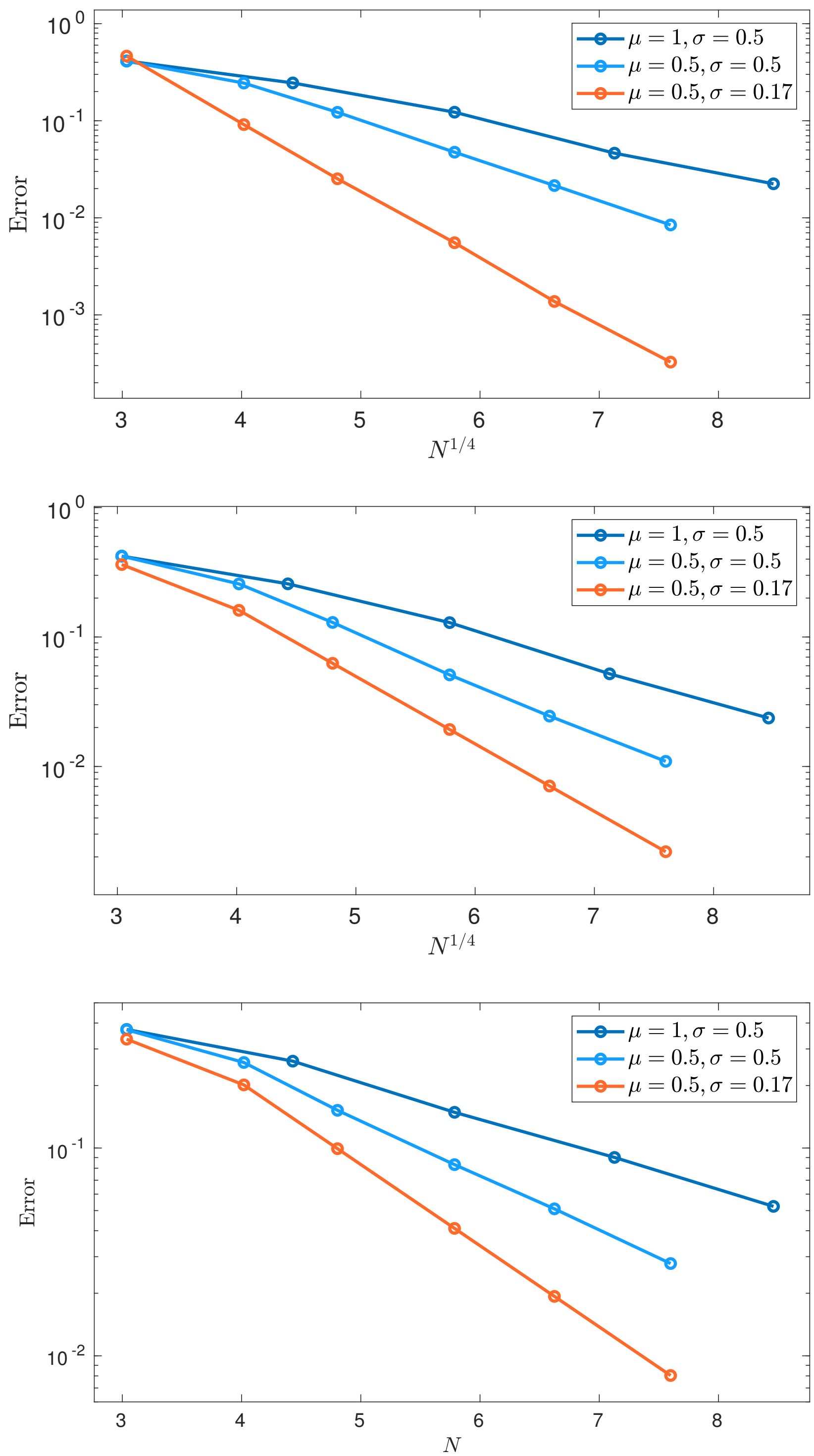

Figure 22: Convergence of hp approximations to u for problem (1) with $f_1 = 1$ for different values of $s = 0.25,\ 0.5,\ 0.75$, Example 8.4.

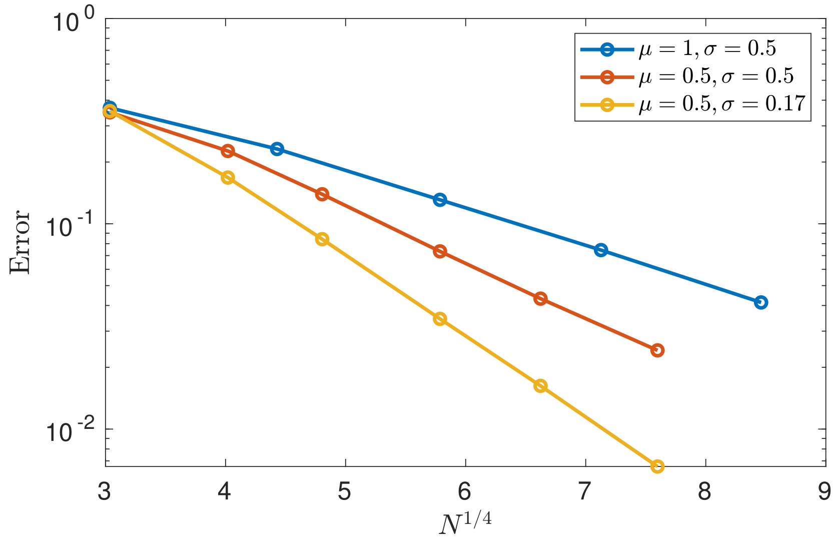

Figure 23: Convergence of hp approximations to u for problem (1) with $f_2 = \sin(2 + 0.2 \cdot (x - y))$ for $s = 0.75$, Example 8.4.

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