Commutator estimates and spectral asymptotics on sub-Riemannian manifolds

Heiko Gimperlein[1]

(joint with Magnus Goffeng[2])

1: Maxwell Institute and Heriot–Watt University, Edinburgh
2: Leibniz Universität Hannover, Germany

Analysis Seminar, Edinburgh
November 25, 2013

Outline

1. Introduction and results
   - Calderón commutator estimate on Heisenberg group
   - Applications: Spectral estimates, noncommutative geometry
2. Heisenberg group and manifolds
   - Basic definitions, operators, function spaces
3. Aspects of proofs

HG, M. Goffeng, *Commutator estimates on sub-Riemannian manifolds and applications*, forthcoming (2013)

Pseudodifferential operators

Fourier transform:

$$\mathcal{F}\phi(\xi) \;=\; \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\,\phi(x)\,dx, \quad \phi\in\mathcal{S}(\mathbb{R}^n)=\{\phi\in C^\infty \mid |x^\beta\partial_x^\beta\phi|<C_{\alpha,\beta}\forall\alpha,\beta\}$$

$\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ isomorphism, $\mathcal{F}\partial_{x_j}\phi=-i\xi_j\mathcal{F}\phi$.
Useful to study differential operators on $\mathbb{R}^n$:

$$P\phi=\textstyle\sum_\alpha a_\alpha(x)\partial_x^\alpha\phi=\mathcal{F}^{-1}_{\xi\to x}\left(\sum_\alpha a_\alpha(x)(-i\xi)^\alpha\,\mathcal{F}\phi(\xi)\right)$$

Pseudodifferential operators: “integral operators” $\supset\{\sum_\alpha a_\alpha(x)\partial_x^\alpha\}$:

$$P\phi(x) \;=\; \mathrm{op}(p)\,\phi(x)=\mathcal{F}^{-1}_{\xi\to x}\left(p(x,\xi)\,\mathcal{F}\phi(\xi)\right), \qquad \phi\in\mathcal{S}(\mathbb{R}^n)$$

$p\sim$ *smooth* rational function: $\quad |\partial_x^\beta\partial_\xi^\alpha p(x,\xi)|\leq C_{\alpha,\beta}(1+|\xi|)^{d-|\alpha|}\;\;\forall\alpha,\beta$
$d=$ order $\in\mathbb{R}$

Pseudodifferential operators (ΨDO)

$$P\phi(x) \ = \ \mathrm{op}(p)\ \phi(x) = \mathcal{F}^{-1}_{\xi\to x}\left(p(x,\xi)\ \mathcal{F}\phi(\xi)\right)$$

- turn PDE into algebra of symbols:

$$\mathrm{op}(p)\ \mathrm{op}(q) = \mathrm{op}\left(pq - i\textstyle\sum_{j=1}^{n}\partial_{\xi_j}p\ \partial_{x_j}q + \dots\right)$$

- on manifold: P ΨDO iff in local coordinates it is a ΨDO on $\mathbb{R}^n$
- developed from integral operator methods:
  Calderón–Zygmund (Acta Math. 1952)
  Kohn–Nirenberg (Comm. Pure Appl. Math. 1965)
  Hörmander (Fields medal), Seeley
  Atiyah–Singer index theorem (Bull. AMS 1963, Ann. Math. 1968)

Main ingredient from harmonic analysis

$P = \sum_{j=1}^{n} a_j(x)\partial_j$, $a_j \in L^\infty(\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz $\implies$

$[P, f]\phi = P(f\phi) - fP\phi = \sum_j a_j(\partial_j f)\phi$ extends to bounded operator on L^p

Calderón commutator estimate

- P pseudodifferential operator of order 1 on $\mathbb{R}^n$
- $f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

$\implies [P, f]$ $L^p(\mathbb{R}^n)$–bounded and $\|[P, f]\|_{L^p \to L^p} \lesssim \|f\|_{Lip}$

The following theorem was apparently not known (HG – Goffeng 2013):

Calderón commutator estimate on Heisenberg group $\mathbb{H}$

- P pseudodifferential operator of order 1 on $\mathbb{H}$
- $f : \mathbb{H} \to \mathbb{R}$ Lipschitz on $\mathbb{H}$

$\implies [P, f]$ $L^p(\mathbb{H})$–bounded and $\|[P, f]\|_{L^p \to L^p} \lesssim \|f\|_{Lip_{\mathbb{H}}}$

Some questions

- What is $\mathbb{H}$?
- What is a pseudodifferential operator on $\mathbb{H}$?
- What about the proof of the commutator estimate? Isn't this trivial?
- Why do we care?
  $\rightsquigarrow$ applications to spectral theory & noncommutative geometry

Application to spectral asymptotics

Theorem (HG–Goffeng 2013)

- M closed sub-Riemannian (Heisenberg) manifold
- T ΨDO of order 0, $a \in C^{0,\alpha}_H$ for some $0 < \alpha \leq 1$

Then $[T,a]$ is a compact operator on $L^2(M)$. More precisely, $[T,a] \in \ell^{\frac{\dim M+1}{\alpha},\infty}(L^2(M))$ and its eigenvalues λ_k satisfy

$$\sup_k |\lambda_k|\, k^{\frac{\alpha}{\dim M+1}} \lesssim \|a\|_{C^{0,\alpha}_H(M)} \,.$$

Remarks

a) The exponent $\frac{\alpha}{\dim M+1}$ is sharp.
b) Complex geometers would be excited to learn that $\lim_{k\to\infty} \lambda_k\, k^{\frac{\alpha}{\dim M+1}}$ exists and compute it.

Application of spectral asymptotics

- X, Y compact, connected, oriented manifolds
- $f : X \to Y$ smooth, $df \neq 0$
- degree of f: $\deg f = \sum_{x=f^{-1}(y)} \mathrm{sign}\ \det df(x)$ homotopy invariant
- analytical formula: α n–form on Y, $\int_Y \alpha = 1$
  $\deg f = \int_X f^*\alpha = \int_X \det\left(\frac{\partial f_i}{\partial x_j}\right)\alpha$
- What about continuous f? Formulas without derivatives.
- Idea of Connes ($X = S^1$): g suitable,
  $\deg f = (-1)^{n+1}\mathrm{index}\ T_{g\circ f} = \dim\ker\ T_{g\circ f} - \dim\mathrm{coker}\ T_{g\circ f}$
- Here: $T_a : H^2(S^1,\mathbb{C}^n) \to H^2(S^1,\mathbb{C}^n)$ Toeplitz operator
  $H^2(S^1,\mathbb{C}^n) = \{\phi \in L^2 :\ \hat{\phi}(n) = 0\ \ \forall n < 0\}$

$$T_a\phi = \Pi_{n\geq 0}(a\phi)$$

Application of spectral asymptotics (ctd.)

- $\deg f = (-1)^{n+1} \text{index } T_{g\circ f} = \dim\ker\ T_{g\circ f} - \dim\operatorname{coker} T_{g\circ f}$
- $T_a : H^2(S^1, \mathbb{C}^n) \to H^2(S^1, \mathbb{C}^n)$ Toeplitz operator

$$T_a\phi = \Pi_{n\geq 0}(a\phi)$$

- analytical index formula (rhs written as $\int_{S^1} \cdots \int_{S^1}$)

$$\text{index } T_a = -\operatorname{tr}_{H^2}\ a^{-1}[\Pi, a][\Pi, a^{-1}] \cdots [\Pi, a][\Pi, a^{-1}]$$

Theorem

- Ω rel. compact strictly pseudoconvex domain in Stein manifold, $\partial\Omega\ C^\infty$
- $H_{\partial\Omega}$ integral kernel of Szegö projection
- $a \in C^{0,\alpha}(\partial\Omega, GL(N))$, $2k+1 > \dim\Omega/\alpha$

$$\Rightarrow \text{index } T_a = -\int_{\partial\Omega^{2k+1}} \operatorname{tr}_{M_N} \prod_{j=0}^{2k}(1 - a(z_{j-1})^{-1}a(z_j))H_{\partial\Omega}(z_{j-1}, z_j)\ .$$

Summary: analytic formula for the degree of $f : \partial\Omega \to GL_N(\mathbb{C})$, $f \in C^{0,\alpha}$

Sub-Riemannian geometry of the Heisenberg group

3–dim. Euclidean space $\mathbb{R}^3$

- $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$
- Lie group $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
- dilations $\lambda x = (\lambda x_1, \lambda x_2, \lambda x_3)$, $\lambda(x+y) = \lambda x + \lambda y$, $|\lambda x| = |\lambda||x|$
- metric $d(x,y) = |x-y| = \inf\{\ell(\gamma) : \gamma$ smooth curve $x \leftrightarrow y\ \}$
- special case of Connes metric:

$$d(x,y) = \sup\{f(x) - f(y) : f \in Lip\ , \|[\nabla, f]\|_{L^2 \to L^2} = \|\nabla f\|_{L^\infty} \leq 1\ \}$$

- Riemannian manifolds locally look like this (mod curvature, anisotropic rescaling)
- Laplacian $\Delta = -\partial_1^2 - \partial_2^2 - \partial_3^2$ elliptic

Sub-Riemannian geometry of the Heisenberg group

3–dim. Heisenberg group $\mathbb{H}$

- as manifold $\mathbb{H} = \mathbb{R}^3$
- $|x|_H = (x_1^4 + x_2^4 + x_3^2)^{1/4}$ Koranyi gauge
- Lie group $x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1))$
- dilations $\lambda \cdot x = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$, $\lambda \cdot (x \cdot y) = \lambda \cdot x + \lambda \cdot y$, $|\lambda x|_H = |\lambda||x|_H$
- Consider bundle $H_{\mathbf{0}} = \mathbb{R}^2 \times \{0\}$, $H_x = x \cdot H_0$.
- Carnot–Caratheodory metric
  $d(x,y) = \inf\{\ell(\gamma) : \gamma$ smooth curve $x \leftrightarrow y$ s.t. $\dot\gamma(t) \in H_{\gamma(t)}\ \forall t\ \}$
  (Chow's Theorem: Such γ exist.)
- Locally $d(x,y) \simeq |x \cdot y^{-1}|_H$.

Sub-Riemannian geometry of $\mathbb{H}$ (ctd.)

3–dim. Heisenberg group $\mathbb{H}$

- special case of Connes metric? $\exists$ operator D s.t.

$$d(x,y) = \sup\{f(x) - f(y) : f \in Lip\ , \|[D,f]\|_{L^2 \to L^2} \leq 1\ \}\ ?$$

- Sub–Laplacian $\Delta_H \sim X_1^* X_1 + X_2^* X_2$ sub–elliptic, where

$$X_1 = \partial_1 - \frac{1}{2} x_2 \partial_3,\ X_2 = \partial_2 + \frac{1}{2} x_1 \partial_3,\ X_3 = \partial_3$$

Hörmander's Theorem: $\{X_j\}_{j=1}^s$ vector fields on $\mathbb{R}^n$ s.t. $\exists d < \infty \forall x$

$$\mathbb{R}^n = \{X_1(x), \dots, X_s(x), [X_1, X_2](x), \dots, \text{commutators up to order d}\ \}\ .$$

Then $\Delta = \sum_{j=1}^s X_j^* X_j$ is sub–elliptic: $\phi \in D'$, $\Delta\phi \in C^\infty \implies \phi \in C^\infty$.

Heisenberg manifolds

Definition

A Heisenberg manifold is a closed, compact Riemannian manifold M endowed with a subbundle $H \subset TM$ of codimension 1 s.t. $TM = H + [H, H]$.

(3–dim.) Heisenberg manifolds locally look like $(\mathbb{H}, H)$ (mod curvature, anisotropic rescaling)

Key examples: complex analysis and contact manifolds

a) $N = \{\varrho < 0\} \subset \mathbb{C}^n$ strictly pseudoconvex, i.e. $i\bar{\partial}\partial\varrho|_{\partial N} > 0$, $M = \partial N$, $H = \ker d^c\varrho$

b) more general: M contact manifold, i.e. $\dim\ M = 2n - 1$, 1–form θ s.t. $\theta \wedge d\theta^n$ non–degenerate, $H = \ker d\theta$

The Sub-Laplacian Δ_H

- (M, H) Heisenberg manifold, $H = \text{span}\{X_1, \dots, X_N\}$
- $\Delta_H = \sum_{j=1}^N X_j^* X_j$ formally on $\phi \in C_c^\infty(M)$
- Self-adjoint closure:
  Bilinear form $\mathfrak{q}_H(\phi) = \sum_{j=1}^N \int_M |X_j\phi|^2 \, dx$ on $C_c^\infty(M)$ closable
  domain of form closure

$$\text{Dom}(\mathfrak{q}_H) = \{\phi \in L^2(M) : X_j\phi \in L^2(M) \forall j\} =: W_H^1(M)$$

  Denote by Δ_H associated self-adjoint operator
- Δ_H compact resolvent $\Rightarrow$ sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$
- Weyl's law / spectral asymptotics (Beals–Greiner–Stanton '87):
  $\lambda_k \, k^{-\frac{2}{\dim M+1}} \to \nu_0$ for $k \to \infty$

Application to noncommutative geometry

Recall that on a closed, compact Riemannian manifold:

$$d(x,y) = \sup\{f(x) - f(y) : f \in Lip\ , \|[\nabla, f]\|_{L^2 \to L^2} = \|\nabla f\|_{L^\infty} \leq 1\ \}$$

In NCG one studies spectral triples: $(\mathcal{A}, \mathcal{H}, D : \mathcal{D} \subset \mathcal{H} \to \mathcal{H})$.
$\mathcal{A}$ is an algebra of operators on the Hilbert space $\mathcal{H}$,
D abstract Fredholm operator, order 1, compact resolvent.
Think $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(M, \mathcal{S})$
$D =$ Dirac-type operator $=$ operator s.t. $D^2 - \Delta_M$ order 1.
$S(\mathcal{A}) =$ positive linear functionals on $\mathcal{A}$ of norm 1. Here $= M$.

Connes metric on state space $S(\mathcal{A})$

$$d_D(\omega, \omega') := \sup\{|\omega(a) - \omega'(a)| : a \in \mathcal{A} :\ \|[D, a]\|_{\mathcal{L}(\mathcal{H})} \leq 1\}$$

For $(\mathcal{A}, \mathcal{H}, D) = (C^\infty, L^2, \text{Dirac})$, ω, ω' point evaluations in x, y:
$\Rightarrow d_D(\omega, \omega') = d(x, y)$ Riemannian distance.

Application to noncommutative geometry (ctd.)

(M, H) closed, compact Heisenberg manifold.
Old question: Is there $(\mathcal{A}, \mathcal{H}, D)$ s.t. $d_D(\omega, \omega') = d(x, y)$?
Dirac operators constructed from X_j don't do it – infinite dimensional kernel.

Theorem (special case)

- D_H horizontal Dirac
- S any operator $\sim$ projection onto $\ker D_H$
- $D = D_H + \theta S$

Then $(C^\infty(M), L^2(M, S^H M), D_\theta)$ spectral triple of same metric dimension as (M, H) s.t.

$$(M, d_{D_\theta}) \to (M, d) \quad \text{in Gromov-Hausdorff distance as} \quad \theta \to 0.$$

Pseudodifferential operators on $U \subset \mathbb{H}$

$$X_1 = \partial_1 - \frac{1}{2}x_2\partial_3,\ X_2 = \partial_2 + \frac{1}{2}x_1\partial_3,\ X_3 = \partial_3$$

$$\sigma_1(x,\xi) = -i\xi_1 + \frac{i}{2}x_2\xi_3,\ \sigma_2(x,\xi) = -i\xi_2 - \frac{i}{2}x_1\xi_3,\ \sigma_3(x,\xi) = -i\xi_3$$

- $d \in \mathbb{C}$, $S_d(U \times \mathbb{H}) =$

  $\{p \in C^\infty(U\times\mathbb{H}\setminus\{0\}) : p(x,\lambda\cdot\xi) = \lambda^d p(x,\xi) \quad \forall\lambda > 0,\ (x,\xi) \in U\times\mathbb{H}\,\}$
- $S^d(U \times \mathbb{H}) = \{p \in C^\infty(U \times \mathbb{H}) :\ \exists p_k \in S_{d-k}(U \times \mathbb{H})\ \forall N\ \forall K \Subset U$

  $\left|\partial_x^\alpha \partial_\xi^\beta \left(p - \sum_{k=0}^N p_k\right)(x,\xi)\right| \leq C_{\alpha,\beta,K,N}|\xi|_H^{\mathrm{Re}\ d-\langle\beta\rangle-N},\quad \forall x \in K,\ |\xi|_H \geq 1\}$
- $P\phi(x)\ =\ \mathrm{op}(p)\ \phi(x) = \mathcal{F}^{-1}_{\xi\to x}\left(p(x,\sigma(x,\xi))\ \mathcal{F}\phi(\xi)\right)$
- on manifold: P ΨDO iff in local coordinates it is a ΨDO on $\mathbb{H}$

Pseudodifferential operators on $U \subset \mathbb{H}$ (ctd.)

$$\sigma_1(x,\xi) = -i\xi_1 + \frac{i}{2}x_2\xi_3,\ \sigma_2(x,\xi) = -i\xi_2 - \frac{i}{2}x_1\xi_3,\ \sigma_3(x,\xi) = -i\xi_3$$

$$P\phi(x) \ = \ \mathrm{op}(p)\ \phi(x) = \mathcal{F}^{-1}_{\xi\to x}\left(p(x,\sigma(x,\xi))\ \mathcal{F}\phi(\xi)\right)$$

Properties

- Δ_H elliptic ΨDO of order 2, Szegö projection order 0
- PDEs on $M \rightsquigarrow$ symbol algebra, order-0 op.'s bounded on $L^p(M)$
- generalized Calderon–Zygmund operators: integral kernel $K_P(x,y) = k_P(x, x-y)$ in local coordinates satisfies

$$|k_P(x,x-y)| \lesssim d(x,y)^{-\dim\ M - d}$$
$$|X_{j,x}k_P(x,x-y)| \lesssim d(x,y)^{-\dim\ M - d - 1}$$
$$|X_{j,y}k_P(x,x-y)| \lesssim d(x,y)^{-\dim\ M - d - 1}$$

- Best constants $\|K_P\|_{CZ}$

Function spaces

Function spaces

- $C^{k,\alpha}(M)$ using Heisenberg metric & X_j
- $W^s_H(M) = (1+\Delta_H)^{-s/2}\, L^2(M)$
- Littlewood–Paley decomposition based on Koranyi gauge $|\cdot|_H$ $\rightsquigarrow HB^s_{p,q}$
- cf. Bahouri–Fermanian-Kammerer–Gallagher (Asterisque 2012), we need sharp estimates!
- qualitative properties (interpolation, embedding thms.) as on $\mathbb{R}^n$
- *quantitative* estimates often only known with loss of derivatives

Order 1 to order 0

Calderón commutator estimate $\Rightarrow$

For any T ΨDO of order 0, $f \in Lip_H$

$$\|[T,f]\|_{L^2 \to W^1_H} \lesssim \|f\|_{Lip_H} \ .$$

Writing $[T,f] = Id \circ [T,f] : L^2 \to W^1_H \to L^2$, the estimate

$$\sup_k |\lambda_k([T,f])| \ k^{\frac{1}{\dim \ M+1}} \lesssim \|f\|_{Lip_H}$$

follows from

- the spectral asymptotics of Δ_H,

$$\lambda_k(\Delta_H) \ k^{-\frac{2}{\dim \ M+1}} \to \nu_0 \ ,$$

- $\ell^{\dim \ M+1,\infty}(L^2(M))$ is an ideal under composition .

Order 1 to order 0

Calderón commutator estimate $\Rightarrow$

For any T ΨDO of order 0, $f \in Lip_H$

$$\|[T, f]\|_{L^2 \to W^1_H} \lesssim \|f\|_{Lip_H} \ .$$

Proof: May reduce to T s.t. $T^2 - 1 = T^* - T = 0$.

$$D_T := \begin{pmatrix} 0 & \sqrt{\Delta_H} T \\ T\sqrt{\Delta_H} & 0 \end{pmatrix} \rightsquigarrow D_T^2 = \begin{pmatrix} \Delta_H & 0 \\ 0 & T\Delta_H T \end{pmatrix} .$$

Thus $1 + D_T^2 > 0$.

$$\widetilde{T} := D_T(1 + D_T^2)^{-1/2} = \begin{pmatrix} 0 & \sqrt{\Delta_H}(1+\Delta_H)^{-1/2} T \\ T\sqrt{\Delta_H}(1+\Delta_H)^{-1/2} & 0 \end{pmatrix} .$$

Note:

$$\widetilde{T} - \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\sqrt{\Delta_H}(1+\Delta_H)^{-1/2} - 1)T \\ T(\sqrt{\Delta_H}(1+\Delta_H)^{-1/2} - 1) & 0 \end{pmatrix}$$

Order 1 to order 0

Calderón commutator estimate $\Rightarrow$

For any T ΨDO of order 0, $f \in Lip_H$

$$\|[T,f]\|_{L^2 \to W^1_H} \lesssim \|f\|_{Lip_H} .$$

$$\widetilde{T} := D_T(1+D_T^2)^{-1/2} = \begin{pmatrix} 0 & \sqrt{\Delta_H}(1+\Delta_H)^{-1/2}T \\ T\sqrt{\Delta_H}(1+\Delta_H)^{-1/2} & 0 \end{pmatrix} .$$

Note:

$$\widetilde{T} - \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\sqrt{\Delta_H}(1+\Delta_H)^{-1/2} - 1)T \\ T(\sqrt{\Delta_H}(1+\Delta_H)^{-1/2} - 1) & 0 \end{pmatrix}$$

Definition of W^1_H: $\widetilde{T} - \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} : L^2 \to W^1_H$ bounded

Order 1 to order 0

Calderón commutator estimate ⇒

For any T ΨDO of order 0, $f \in Lip_H$

$$\|[T,f]\|_{L^2 \to W^1_H} \lesssim \|f\|_{Lip_H} \ .$$

Definition of W^1_H: $\widetilde{T} - \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} : L^2 \to W^1_H$ bounded

Operator inequality: For $f \in Lip_{CC}(M)$ with values in $i\mathbb{R}$,

$$-\|[D_T,f]\|_{L^2 \to L^2}(1+D_T^2)^{-1/2} \leq [\tilde{T},f] \leq \|[D_T,f]\|_{L^2 \to L^2}(1+D_T^2)^{-1/2}.$$

$$\Rightarrow \quad \|[\tilde{T},f]\|_{L^2 \to W^1_H}$$
$$\leq \|(1+D_T^2)^{-1/2}\|_{L^2 \to W^1_H} \|[D_T,f]\|_{L^2 \to L^2} \leq_{CCE} C\|f\|_{Lip_H}.$$

From Lip_H to $C_H^{0,\alpha}$

Conclusion

For any T ΨDO of order 0, $a \mapsto [T, a]$ is bounded

- $Lip_H(M) \to \ell^{\dim\ M+1,\infty}(L^2(M))$,
- (easy) $C^0(M) \to \mathcal{K}(L^2(M))$

Interpolation $\rightsquigarrow C^{0,\alpha}(M) \to \ell^{\frac{\dim\ M+1}{\alpha},\infty}(L^2(M))$, i.e.

$$\sup_k |\lambda_k([T, a])|\ k^{\frac{\alpha}{\dim\ M+1}} \lesssim \|a\|_{C_H^{0,\alpha}(M)}\ .$$

Pseudodifferential calculus shows $C_H^k(M) \to \ell^{\dim\ M+1,\infty}(L^2(M))$ for large k.

A *Tb* theorem by Hytönen–Martikainen

Hytönen–Martikainen (2012)

Let T be an L^2-bounded integral operator with Calderón–Zygmund kernel K_T, $b_1, b_2 \in L^\infty(M)$ with $\mathrm{Re}\, b_1, \mathrm{Re}\, b_2 > c > 0$, $\kappa, \Lambda > 1$ and $S : (0,1] \to (0,\infty)$. Then

$$\|T\|_{\mathcal{L}(L^2(M))} \lesssim \|Tb_1\|_{BMO^2_\kappa} + \|T^* b_2\|_{BMO^2_\kappa} + \|b_2 T b_1\|_{WBP_{\Lambda,S}} + \|K_T\|_{CZ} \,.$$

Here $\|T\|_{WBP_{\Lambda,S}}$ is the best constant C s.t. $\langle T\chi_{B,\varepsilon}, \chi_{B,\varepsilon}\rangle \leq CS(\varepsilon)|\Lambda \cdot B|$ for fixed cut–off function $\chi_B \leq \chi_{B,\varepsilon} \leq \chi_{(1+\varepsilon)B}$.

The basic idea in our proof of the Calderón commutator estimate is to
a) show that $T = [D, f]$ bounded,
b) deduce operator norm $\lesssim \|f\|_{Lip_H}$ by estimating the right hand side.

Conclusions & Outlook

- Harmonic analysis on $\mathbb{H}$ is useful in other areas
- Calderón commutator estimate for $[D^1, f]$ implies sharp spectral estimates for $[T^0, a]$
- Elliptic techniques of Birman–Solomyak, Sukochev et al. should allow to prove existence / compute

$$\lim_{k\to\infty} \lambda_k([T,a])\, k^{\frac{\alpha}{\dim\ M+1}} = ?$$

and certain regularized integrals in complex analysis.
- But: Need sharp estimates on $\mathbb{H}$!