Boundary elements for contact problems: Stabilization, hp-methods, dynamics

Heiko Gimperlein
(joint with L. Banz, A. Issaoui, F. Meyer, C. Özdemir and E. P. Stephan)

Heriot–Watt University, Edinburgh, UK

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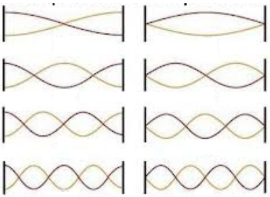
- Basics of contact and boundary element methods
- Mixed formulation of contact problems
- Stabilized hp-methods in elasticity:
  a priori and a posteriori error estimates,
  adaptive algorithms
- Dynamic contact for the wave equation:
  a priori error estimates

L. Banz, HG, A. Issaoui, E. P. Stephan, Stabilized mixed hp-BEM for frictional contact problems in linear elasticity, Numer. Math. 135 (2017), 217-263.

HG, F. Meyer, C. Özdemir, E. P. Stephan, Time domain boundary elements for dynamic contact problems, CMAME 333 (2018), 147-175.

Boundary conditions

Neumann and Dirichlet:



Contact: Signorini (= nonpenetration, $\perp$ wall) and friction ($\parallel$ wall)

Toy problem for Laplacian

$$\begin{aligned} U &: \Omega_x \to \mathbb{R}, \\ -\Delta U &= 0 \quad \text{in } \Omega. \end{aligned}$$

Dirichlet-Neumann operator on $\Gamma := \partial\Omega$

$$\mathcal{S}(U|_\Gamma) = \frac{\partial U}{\partial \nu}\Big|_\Gamma$$

contact boundary conditions on $\Gamma_C \subset \Gamma$ ($U = 0$ on $\Gamma \setminus \overline{\Gamma_C}$)

$$\begin{cases} U \leq g \ , \ \frac{\partial U}{\partial \nu} \leq 0 \ , \\ U < g \implies \frac{\partial U}{\partial \nu} = 0 \ . \end{cases}$$

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Variational inequality: Find
$0 \leq u = U|_G \in K := \{w \in \widetilde{H}^{1/2}(\Gamma_C) : w \leq g\}$ such that

$$\langle \mathcal{S}u, v - u \rangle_{\Gamma_C} \geq 0 \qquad \forall\, v \in K \ .$$

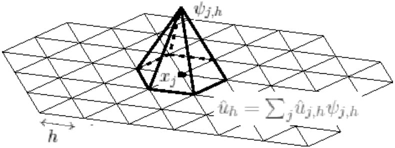
Dirichlet–Neumann operator – Why?

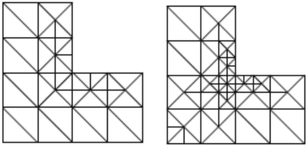
$$\mathcal{S}(U|_\Gamma) = \frac{\partial U}{\partial \nu}\Big|_\Gamma$$

Key: Reduces bilinear form of the Laplacian from Ω to $\Gamma = \partial\Omega$:

$$\int_\Omega |\nabla U|^2 = \langle \partial_\nu U, U \rangle_\Gamma = \langle \mathcal{S}(U|_\Gamma)), U \rangle_\Gamma$$

Numerical approximation on Γ: **B**oundary **E**lements





Dirichlet–Neumann operator – Why?

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Key: Reduces bilinear form of the Laplacian from Ω to $\Gamma = \partial\Omega$:

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$+$ $\dim\, \partial\Omega = n-1$, $\partial\Omega$ bounded

$+$ hp-boundary element methods: exponential convergence

$+$ compression/preconditioning: dense matrices ok

$-$ homogeneous linear equations

$-$ dense matrices: storage, time $\sim$ (DOF)^2, need optimization

Dirichlet–Neumann operator / Multi-layer potentials

$$\int_\Omega |\nabla U|^2 = \langle \partial_\nu U, U\rangle_\Gamma = \langle \mathcal{S}(U|_\Gamma)), U\rangle_\Gamma$$

$H^{\frac{1}{2}}(\Gamma)$–coercive: $\langle \mathcal{S}(U|_\Gamma), U\rangle_\Gamma \geq \alpha \|U\|^2_{H^{\frac{1}{2}}(\Gamma)}$

Contact problem is coercive variational inequality on Γ_C:
Find $0 \leq u \in K := \{w \in \widetilde{H}^{1/2}(\Gamma_C) : w \leq g\}$ such that

$$\langle \mathcal{S}u, v-u\rangle_{\Gamma_C} \geq 0 \qquad \forall\ v \in K\ .$$

Dirichlet–Neumann operator / Multi-layer potentials

$$\int_\Omega |\nabla U|^2 = \langle \partial_\nu U, U\rangle_\Gamma = \langle \mathcal{S}(U|_\Gamma)), U\rangle_\Gamma$$

Contact problem is coercive variational inequality on Γ_C:
Find $0 \leq u \in K := \{w \in \widetilde{H}^{1/2}(\Gamma_C) : w \leq g\}$ such that

$$\langle \mathcal{S}u, v - u\rangle_{\Gamma_C} \geq 0 \qquad \forall\ v \in K\ .$$

Mixed formulation of contact problem on Γ_C: (Contact forces $\lambda = -\mathcal{S}u$)
Find $(u, \lambda) \in \widetilde{H}^{1/2}(\Gamma_C) \times M^+$ s.t.

$$\begin{aligned} \langle \mathcal{S}u, v\rangle_{\Gamma_C} + \langle \lambda, v\rangle_{\Gamma_C} &= 0 && \forall v \in \tilde{H}^{1/2}(\Gamma_C) \\ \langle u, \mu - \lambda\rangle_{\Gamma_C} &\leq \langle g, \mu - \lambda\rangle_{\Gamma_C} && \forall \mu \in M^+). \end{aligned}$$

$$M^+ = \left\{\mu \in \tilde{H}^{-1/2}(\Gamma_C) : \langle \mu, v\rangle_{\Gamma_C} \leq 0 \ \ \forall v \in \tilde{H}^{1/2}(\Gamma_C), v \leq 0\right\}$$

Dirichlet–Neumann operator / Multi-layer potentials

Mixed formulation of contact problem on Γ_C: (Contact forces $\lambda = -\mathcal{S}u$)
Find $(u, \lambda) \in \widetilde{H}^{1/2}(\Gamma_C) \times M^+$ s.t.

$$\langle \mathcal{S}u, v\rangle_{\Gamma_C} + \langle \lambda, v\rangle_{\Gamma_C} = 0 \qquad \forall v \in \tilde{H}^{1/2}(\Gamma_C)$$

$$\langle u, \mu - \lambda\rangle_{\Gamma_C} \leq \langle g, \mu - \lambda\rangle_{\Gamma_C} \qquad \forall \mu \in M^+).$$

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Galerkin approximation using layer potentials:
$\mathcal{S} = \mathcal{W} + (1 - \mathcal{K}')\mathcal{V}^{-1}(1 - \mathcal{K})$

$$\mathcal{V}\phi(x) = \int_\Gamma k(x,y)\phi(y)\ ds_y\ , \qquad k(x,y) = \begin{cases} -\frac{1}{2\pi}\log|x-y|, & n = 2 \\ \frac{1}{4\pi}\frac{1}{|x-y|}, & n = 3. \end{cases}$$

$\mathcal{W}$, $\mathcal{K}$, $\mathcal{K}'$ similar using normal derivatives of k.

High–order methods: Mixed hp–BEM in elasticity

- displacement u, stress $\sigma = \sigma(u)$
- non–penetration (Signorini) condition on normal components

$$\sigma_n \leq 0, \; u_n \leq g, \; \sigma_n(u_n - g) = 0$$

- friction (Tresca) condition on tangential components

$$|\sigma_t| \leq \mathcal{F}, \; \sigma_t u_t + \mathcal{F}\,|u_t| = 0$$

- Dirichlet Γ_D, Neumann Γ_N, $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$, $\overline{\Gamma}_\Sigma := \overline{\Gamma}_N \cup \overline{\Gamma}_C$

Mixed formulation ($\lambda = -\sigma(u)n$):
Find $(u, \lambda) \in \tilde{H}^{1/2}(\Gamma_\Sigma) \times M^+(\mathcal{F})$ s.t.

$$\langle Su, v\rangle_{\Gamma_\Sigma} + \langle \lambda, v\rangle_{\Gamma_C} = \langle f, v\rangle_{\Gamma_N} \qquad \forall v \in \tilde{H}^{1/2}(\Gamma_\Sigma)$$

$$\langle u, \mu - \lambda\rangle_{\Gamma_C} \leq \langle g, \mu_n - \lambda_n\rangle_{\Gamma_C} \qquad \forall \mu \in M^+(\mathcal{F}).$$

$$M^+(\mathcal{F}) = \left\{\mu \in \tilde{H}^{-1/2}(\Gamma_C) : \langle \mu, v\rangle_{\Gamma_C} \leq \langle \mathcal{F}, |v_t|\rangle_{\Gamma_C} \, \forall v \in \tilde{H}^{1/2}(\Gamma_\Sigma), v_n \leq 0\right\} \quad (1)$$

Mixed formulation = saddle–point problem

Find $(u, \lambda) \in \tilde{H}^{1/2}(\Gamma_\Sigma) \times M^+(\mathcal{F})$ s.t.

$$\langle Su, v\rangle_{\Gamma_\Sigma} + \langle \lambda, v\rangle_{\Gamma_C} = \langle f, v\rangle_{\Gamma_N} \qquad \forall v \in \tilde{H}^{1/2}(\Gamma_\Sigma)$$
$$\langle u, \mu - \lambda\rangle_{\Gamma_C} \leq \langle g, \mu_n - \lambda_n\rangle_{\Gamma_C} \qquad \forall \mu \in M^+(\mathcal{F}).$$

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Uniquely solvable if the following operator is nondegenerate:

$$\begin{pmatrix} \langle S\cdot, \cdot\rangle_{\Gamma_\Sigma} & \langle \cdot, \cdot\rangle_{\Gamma_C} \\ \langle \cdot, \cdot\rangle_{\Gamma_C} & 0 \end{pmatrix}$$

inf–sup condition necessary and sufficient:

$$\tilde{\beta} \, \|\mu\|_{\tilde{H}^{-1/2}(\Gamma_C)} \leq \sup_{v \in \tilde{H}^{1/2}(\Gamma_\Sigma) \setminus \{0\}} \frac{\langle \mu, v\rangle_{\Gamma_C}}{\|v\|_{\tilde{H}^{1/2}(\Gamma_\Sigma)}} \qquad \forall \mu \in \tilde{H}^{-1/2}(\Gamma_C) \; .$$

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^{+}_{kq}(\mathcal{F})$ s.t.

$$\left\langle S_{hp} u^{hp}, v^{hp} \right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp} \right\rangle_{\Gamma_C} = \left\langle f, v^{hp} \right\rangle_{\Gamma_N} \qquad \forall v^{hp} \in \mathcal{V}_{hp}$$
$$\left\langle \mu^{kq} - \lambda^{kq}, u^{hp} \right\rangle_{\Gamma_C} \leq \left\langle g, \mu^{kq}_n - \lambda^{kq}_n \right\rangle_{\Gamma_C} \qquad \forall \mu^{kq} \in M^{+}_{kq}(\mathcal{F}) \; .$$

Inf–sup needs: Two meshes on the boundary $\mathcal{T}_h$, $\hat{\mathcal{T}}_k$

$$\mathcal{V}_{hp} = \{ v^{hp} \in C^0(\Gamma_\Sigma) \cap \tilde{H}^{1/2}(\Gamma_\Sigma) : v^{hp}|_E \in [\mathbb{P}_{p_E}]^2 \,, \; v_{hp} = 0 \text{ at } \partial\Gamma_\Sigma \}$$

$$M^{+}_{kq}(\mathcal{F}) = \{ \mu^{kq} : \mu^{kq}|_E \in [\mathbb{P}_{q_E}]^2 \,, \; \mu^{kq}_n \geq 0, \; |\mu^{kq}_t(x)| \leq \mathcal{F}(x) \text{ for } x \in G_{kq} \}$$

G_{kq} Gauss points, $M^{+}_{kq}(\mathcal{F}) \not\subset M^{+}(\mathcal{F})$.

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ s.t.

$$\left\langle S_{hp} u^{hp}, v^{hp} \right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp} \right\rangle_{\Gamma_C} = \left\langle f, v^{hp} \right\rangle_{\Gamma_N} \qquad \forall v^{hp} \in \mathcal{V}_{hp}$$

$$\left\langle \mu^{kq} - \lambda^{kq}, u^{hp} \right\rangle_{\Gamma_C} \leq \left\langle g, \mu^{kq}_n - \lambda^{kq}_n \right\rangle_{\Gamma_C} \qquad \forall \mu^{kq} \in M^+_{kq}(\mathcal{F}) \ .$$

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Discretization uniquely solvable if the following operator is nondegenerate:

$$\begin{pmatrix} \langle S_{hp} \cdot, \cdot \rangle_{\Gamma_\Sigma} & \langle \cdot, \cdot \rangle_{\Gamma_C} \\ \langle \cdot, \cdot \rangle_{\Gamma_C} & 0 \end{pmatrix}$$

True when $\mathcal{T}_h$ much finer than $\hat{\mathcal{T}}_k$ (Babuska, Schröder, ...)

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ s.t.

$$\left\langle S_{hp}u^{hp}, v^{hp}\right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp}\right\rangle_{\Gamma_C} = \left\langle f, v^{hp}\right\rangle_{\Gamma_N} \qquad \forall v^{hp} \in \mathcal{V}_{hp}$$

$$\left\langle \mu^{kq} - \lambda^{kq}, u^{hp}\right\rangle_{\Gamma_C} \leq \left\langle g, \mu^{kq}_n - \lambda^{kq}_n\right\rangle_{\Gamma_C} \qquad \forall \mu^{kq} \in M^+_{kq}(\mathcal{F}) \ .$$

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True when $\mathcal{T}_h$ much finer than $\hat{\mathcal{T}}_k$ (Babuska, Schröder, ...)
Stabilization: Instead of using two meshes $\mathcal{T}_h$, $\hat{\mathcal{T}}_k$, add consistent terms to discretized problem. We adapt an approach of Barbosa-Hughes to BEM.

Discretization

Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ s.t. for all $(v^{hp}, \mu^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$

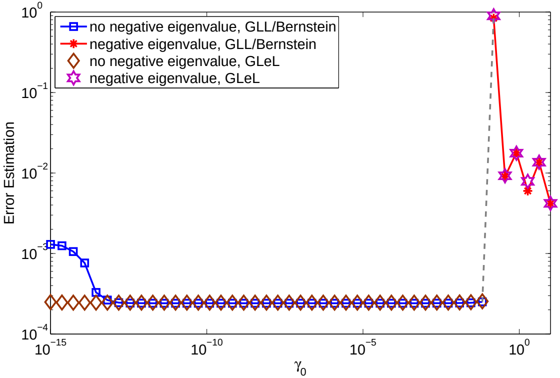
$$\langle S_{hp}u^{hp}, v^{hp}\rangle_{\Gamma_\Sigma} + \langle \lambda^{kq}, v^{hp}\rangle_{\Gamma_C} - \langle \gamma(\lambda^{kq} + S_{hp}u^{hp}), S_{hp}v^{hp}\rangle_{\Gamma_C} = \langle f, v^{hp}\rangle_{\Gamma_N}$$
$$\langle \mu^{kq} - \lambda^{kq}, u^{hp}\rangle_{\Gamma_C} - \langle \gamma(\mu^{kq} - \lambda^{kq}), \lambda^{kq} + S_{hp}u^{hp}\rangle_{\Gamma_C} \leq \langle g, \mu^{kq}_n - \lambda^{kq}_n\rangle_{\Gamma_C}$$

with $\gamma = \gamma_0 \frac{h}{p^2}$. $\lambda = -Su \rightsquigarrow$ new terms ~ 0, consistency error $S \neq S_{hp}$.

$$M^+_{kq}(\mathcal{F}) = \{\mu^{kq} : \mu^{kq}|_E \in [\mathbb{P}_{q_E}]^2,\ \mu^{kq}_n \geq 0,\ |\mu^{kq}_t(x)| \leq \mathcal{F}(x) \text{ for } x \in G_{kq}\}$$

Theorem: If $\gamma_0 > 0$ is small enough, the stabilized discrete problem admits a unique solution.

Error estimator vs. γ_0



uniform discretisation with 256 elements, $p = 1$

Details of numerical experiment later in this talk

Convergence

Main challenges:

- stabilization is not exact: $\lambda = -Su \neq -S_{hp}u$
- nonconforming: $M^+_{kq}(\mathcal{F}) \not\subset M^+(\mathcal{F})$

Theorem (a priori error estimate)

Assume $(u, \lambda) \in H^{1+\alpha}(\Gamma) \times H^{\alpha}(\Gamma_C) \cap C^0(\Gamma_C)$, $\alpha \in [0, \frac{1}{2})$, $h = k$, $p = q + 1$.

$$\rightsquigarrow \|u - u^{hp}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} + \|\gamma^{\frac{1}{2}}(\lambda - \lambda^{hq})\|^2_{L^2(\Gamma_C)} + \|\psi - \psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \lesssim h^{\alpha/2} p^{-\alpha/3}$$

- intricate proof, more complicated rate for arbitrary discretisations
- compare Hild, Lleras, Renard for FEM, rates not optimal
- rate dominated by $S \neq S_{hp}$
- more involved estimates for $h \neq k$, $p \neq q + 1$

Tresca friction: a posteriori error estimate

Theorem

$$
\begin{aligned}
&\|u-u^{hp}\|^2_{\tilde H^{1/2}(\Gamma_\Sigma)} + \|\psi-\psi^{hp}\|^2_{H^{-1/2}(\Gamma)} + \|\lambda^{kq}-\lambda\|^2_{\tilde H^{-1/2}(\Gamma_C)} \\
&\lesssim \sum_{E\in\mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E}\left\|f-S_{hp}u^{hp}\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E}\left\|\lambda^{kq}+S_{hp}u^{hp}\right\|^2_{L^2(E)} \\
&+ \sum_{E\in\mathcal{T}_{h,\Gamma}} h_E\left\|\frac{\partial}{\partial s}\left(V\psi^{hp}-(K+\frac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)} + \left\langle\left(\lambda_n^{kq}\right)^+,\left(g-u_n^{hp}\right)^+\right\rangle_{\Gamma} \\
&+ \left\|\left(g-u_n^{hp}\right)^-\right\|^2_{H^{1/2}(\Gamma_C)} + \left\|\left(\lambda_n^{kq}\right)^-\right\|^2_{\tilde H^{-1/2}(\Gamma_C)} \\
&+ \left\|\left(\left|\lambda_t^{kq}\right|-\mathcal{F}\right)^+\right\|^2_{\tilde H^{-1/2}(\Gamma_C)} - \left\langle\left(\left|\lambda_t^{kq}\right|-\mathcal{F}\right)^-,\left|u_t^{hp}\right|\right\rangle_{\Gamma_C} \\
&- \left\langle\lambda_t^{kq},u_t^{hp}\right\rangle_{\Gamma_C} + \left\langle\left|\lambda_t^{kq}\right|,\left|u_t^{hp}\right|\right\rangle_{\Gamma_C}
\end{aligned}
$$

Solve-mark-refine algorithm for hp-adaptivity

1. Choose: initial mesh $\mathcal{T}_{h,\Gamma}$ and p, parameters $\theta, \delta \in (0,1)$
2. For $k = 0, 1, 2, \ldots$ do
   1. solve discrete mixed problem.
   2. compute local indicators Ξ^2 to current solution.
   3. mark all elements $E \in \mathcal{N} := \operatorname{argmin}\left\{\left|\left\{\hat{\mathcal{N}} \subset \mathcal{T}_{h,\Gamma} : \sum_{E \in \hat{\mathcal{N}}} \Xi^2(E) \geq \theta \sum_{E \in \mathcal{T}_{h,\Gamma}} \Xi^2(E)\right\}\right|\right\}$ for refinement.
   4. decide p or h refinement based on local smoothness, determined by Legendre coefficients

      $$v^{hp}|_E(\Theta_E(x)) = \sum_{j=0}^{p_E} a_i L_i(x), \qquad a_i = \frac{2i+1}{2} \int_{-1}^{1} v^{hp}|_E(\Theta_E(x)) L_i(x)\, dx$$

   5. refine marked elements based on this decision

Numerical example: Tresca friction

$\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$, $\Gamma_C = [-\frac{1}{2}, \frac{1}{2}] \times \{-\frac{1}{2}\}$, $\Gamma_D = [\frac{1}{4}, \frac{1}{2}] \times \{\frac{1}{2}\}$ and
$\Gamma_N = \partial\Omega \setminus (\Gamma_C \cup \Gamma_D)$, $\Gamma_N = \partial\Omega \setminus \Gamma_C \cup \Gamma_D$
Elasticity parameters $E = 500$, $\nu = 0.3$
gap function $g = 1 - \sqrt{1 - \frac{x_1^2}{100}}$, Tresca friction $\mathcal{F} = 0.211 + 0.412x_1$

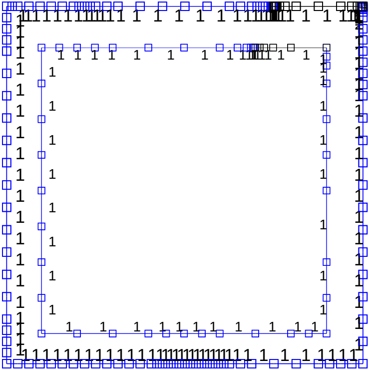
Force on Γ_N:

$$t_{\text{left}} = \begin{pmatrix} -(\frac{1}{2} - x_2)(-\frac{1}{2} - x_2) \\ 0 \end{pmatrix} \qquad \text{on } \left\{-\tfrac{1}{2}\right\} \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right],$$

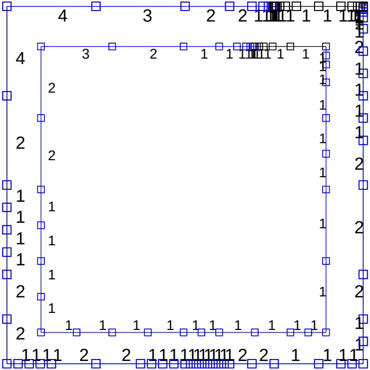
$$t_{\text{top}} = \begin{pmatrix} 0 \\ 20(-\frac{1}{2} - x_1)(-\frac{1}{4} - x_1) \end{pmatrix} \qquad \text{on } \left[-\tfrac{1}{2}, -\tfrac{1}{4}\right] \times \left\{\tfrac{1}{2}\right\},$$

Solution:

- two singular points at interface from Γ_N to Γ_D, worse loss of regularity than from the contact conditions
- on Γ_C long interval with sliding, ($|\sigma_t| = \mathcal{F}$, $u_t = -\alpha\sigma_t$ for $\alpha \geq 0$)



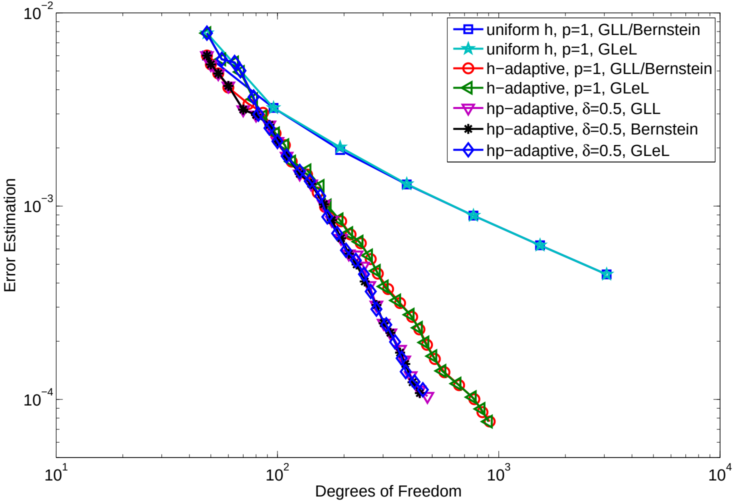
(a) h-adaptive (GLL/Bernstein), mesh nr. 10 (inner), nr. 20 (outer)



(b) hp-adap. (GLL), mesh nr. 10 (inner), nr. 20 (outer)

Figure: Adaptively generated meshes (Tresca friction)

Tresca friction: error of h- and hp-adaptive methods



Coulomb friction

- Tresca friction mathematically nice, but unphysical
- more realistic Coulomb friction: friction threshold $\mathcal{F}$ replaced by $\mathcal{F}|\sigma_n(u)|$
- existence of solution to continuous problem unknown
- in discretization, only set for Lagrange multiplier must be adapted:

$$M_{kq}^{+}(\mathcal{F}\lambda_n^{kq}) = \{\mu^{kq} : \mu^{kq}|_E \in [\mathbb{P}_{q_E}]^2 \,,\ \mu_n^{kq} \geq 0,\ |\mu_t^{kq}(x)| \leq \mathcal{F}\lambda_n^{kq} \ \ (x \in G$$

- a posteriori estimate under regularity assumptions (Hild, Renard, Lleras)

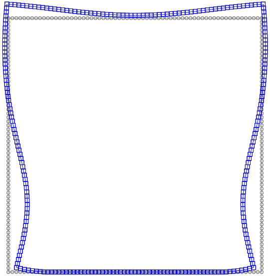
Coulomb friction: a posteriori error estimate

Theorem

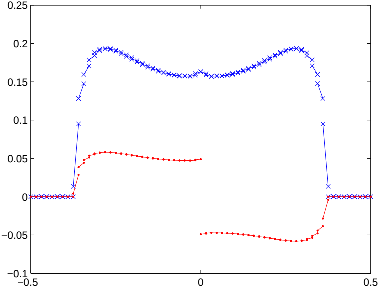
Let $\mathcal{F} \geq 0$ constant, $\lambda_t = \mathcal{F}\lambda_n \xi$, $\xi \in Dir_t(u_t)$, where $Dir_t(u_t)$ subdifferential of $u_t \mapsto |u_t|$, and assume $\mathcal{F}\,\|\xi\|$ is sufficiently small

$$\|u - u^{hp}\|^2_{\tilde{H}^{1/2}(\Gamma_\Sigma)} + \|\psi - \psi^{hp}\|^2_{H^{-1/2}(\Gamma)} + \|\lambda^{kq} - \lambda\|^2_{\tilde{H}^{-1/2}(\Gamma_C)}$$
$$\lesssim \sum_{E \in \mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E} \left\|f - S_{hp}u^{hp}\right\|^2_{L^2(E)} + \sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E} \left\|\lambda^{kq} + S_{hp}u^{hp}\right\|^2_{L^2(E)}$$
$$+ \sum_{E \in \mathcal{T}_{h,\Gamma}} h_E \left\|\frac{\partial}{\partial s}\left(V\psi^{hp} - (K + \tfrac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)} + \left\langle \left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+ \right\rangle$$
$$+ \left\|\left(g - u_n^{hp}\right)^-\right\|^2_{H^{1/2}(\Gamma_C)} + \left\|\left(\lambda_n^{kq}\right)^-\right\|^2_{\tilde{H}^{-1/2}(\Gamma_C)}$$
$$+ \left\|\left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^+\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} - \left\langle \left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^-, \left|(u^{hp})_t\right| \right\rangle$$
$$- \left\langle \lambda_t^{kq}, u_t^{hp} \right\rangle_{\Gamma_C} + \left\langle \left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right| \right\rangle_{\Gamma_C}.$$

Coulomb friction: displacement and forces



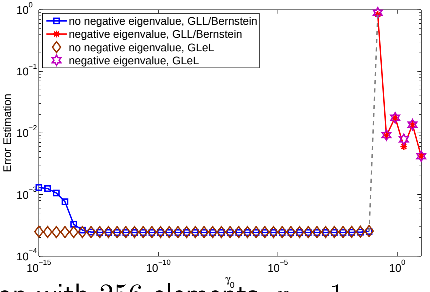
(a) Reference (circle), deformed (square)



(b) λ_n (cross), λ_t (dot)

Figure: Solution of the Coulomb-friction problem, uniform mesh 256 elements, $p = 1$ (GLL/Bernstein)

Coulomb friction: Error estimator vs. γ_0



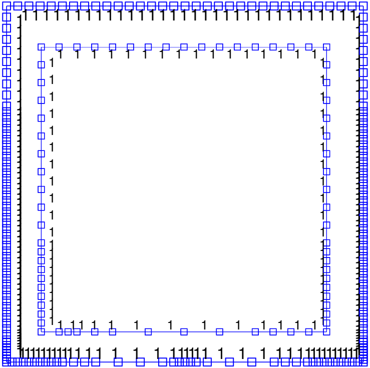
uniform discretisation with 256 elements, $p = 1$

$\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$, $\Gamma_C = [-\frac{1}{2}, \frac{1}{2}] \times \{-\frac{1}{2}\}$, $\Gamma_N = \partial\Omega \setminus \Gamma_C$

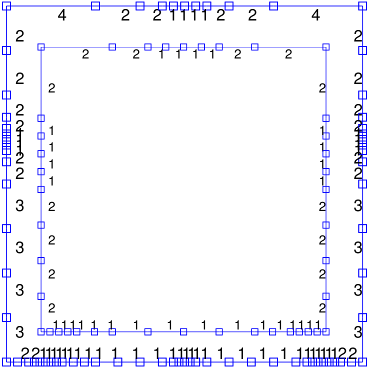
Elasticity parameters $E = 5$, $\nu = 0.45$, friction coefficient 0.3.

$$t_{\text{side}} = \begin{pmatrix} -10\,\mathrm{sign}(x_1)(\frac{1}{2} + x_2)(\frac{1}{2} - x_2)\exp(-10(x_2 + \frac{4}{10})^2) \\ \frac{7}{8}(\frac{1}{2} + x_2)(\frac{1}{2} - x_2) \end{pmatrix}$$

$$t_{\text{top}} = \begin{pmatrix} 0 \\ -\frac{25}{2}(\frac{1}{2} - x_1)^2(\frac{1}{2} + x_1)^2 \end{pmatrix}$$



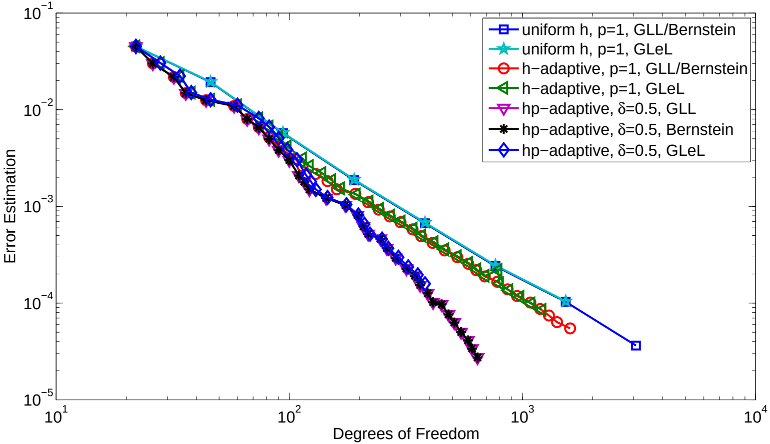
(a) h-adap. (GLL/Bernstein), mesh nr. 16 (in), 25 (out)



(b) hp-adap. (Bernstein), mesh nr. 16 (in), nr. 25 (out)

Figure: Adaptively generated meshes (Coulomb friction)

Coulomb friction: error of h- and hp-adaptive methods



Dynamic contact for wave equation

$$U : \mathbb{R}_t \times \Omega_x^c \to \mathbb{R}$$

$$\partial_t^2 U - \Delta U = 0 \quad \text{in } \mathbb{R} \times \Omega^c, \quad \Omega^c = \mathbb{R}^d \setminus \overline{\Omega},$$

$$U = 0 \quad \text{for } t \leq 0 \ .$$

Real problem:
Lamé equation for tire dynamics – tire does not penetrate road

Key engineering problem, analysis challenging

Dynamic contact for wave equation

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$$\partial_t^2 U - \Delta U = 0 \quad \text{in } \mathbb{R} \times \Omega^c, \quad \Omega^c = \mathbb{R}^d \setminus \overline{\Omega},$$

$$U = 0 \quad \text{for } t \leq 0 \ .$$

Real problem:
Lamé equation for tire dynamics – tire does not penetrate road

Scalar model problem:
wave equation with contact boundary conditions on $G \subset \Gamma := \partial\Omega$

$$\begin{cases} U|_G \geq 0 \ , \ -\frac{\partial U}{\partial \nu}\Big|_G \geq h \ , \\ U|_G > 0 \implies -\frac{\partial U}{\partial \nu}\Big|_G = h \ . \end{cases}$$

Dynamic contact for wave equation

$$\begin{aligned} U : \mathbb{R}_t \times \Omega_x^c &\to \mathbb{R} \\ \partial_t^2 U - \Delta U &= 0 \quad \text{in } \mathbb{R} \times \Omega^c, \quad \Omega^c = \mathbb{R}^d \setminus \overline{\Omega}, \\ U &= 0 \quad \text{for } t \leq 0 \ . \end{aligned}$$

Real problem:
Lamé equation for tire dynamics – tire does not penetrate road

Scalar model problem:
wave equation with contact boundary conditions on $G \subset \Gamma := \partial\Omega$

Recent or ongoing work in FEM: Chouly, Hild, Lleras, Renard, Hauret, Le Tallec, Wohlmuth, . . .

Dirichlet-Neumann operator for wave equation

$$U : \mathbb{R}_t \times \Omega_x \to \mathbb{R}$$

$$\partial_t^2 U - \Delta U = 0 \quad \text{in } \mathbb{R}_t \times \Omega_x^c, \quad \Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$$

$$U = 0 \quad \text{for } t \leq 0 .$$

Dirichlet-Neumann operator on $\Gamma := \partial\Omega$

$$\mathcal{S}(U|_\Gamma) = -\frac{\partial U}{\partial \nu}\Big|_\Gamma$$

Variational inequality in space-time anisotropic Sobolev spaces:

Find $0 \leq u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \widetilde{H}^{\frac{1}{2}}(G))$ such that

$$\langle \mathcal{S}u, v - u\rangle_{\mathbb{R}^+ \times G} \geq \langle h, v - u\rangle_{\mathbb{R}^+ \times G} \qquad \forall\ 0 \leq v \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \widetilde{H}^{\frac{1}{2}}(G)) .$$

Dirichlet-Neumann operator for wave equation

$$U : \mathbb{R}_t \times \Omega_x \to \mathbb{R}$$
$$\partial_t^2 U - \Delta U = 0 \quad \text{in } \mathbb{R}_t \times \Omega_x^c, \quad \Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$$
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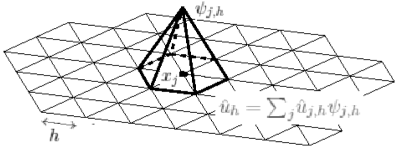
$\Omega = \mathbb{R}^d_-$, $G \subset \Gamma$ bounded polygon.

Existence of solutions for smooth h: Lebeau – Schatzmann '84, students of Eskin, including Cooper '99.

Space-time Galerkin discretization (tensor products)

- $\Gamma = \cup_{i=1}^{M} \Gamma_i$ (quasi-uniform) triangulation
- V_h^p piecewise polynomial functions of degree p on $\Gamma = \cup_{i=1}^{M} \Gamma_i$ (continuous if $p \geq 1$)
- $[0, T) = \cup_{n=1}^{L} [t_{n-1}, t_n)$, $t_n = n(\Delta t)$
- $V_{\Delta t}^q$ piecewise polynomial functions of degree q in time (continuous and vanishing at $t = 0$ if $q \geq 1$)
- tensor products in space-time: $V_{h,\Delta t}^{p,q} = V_h^p \otimes V_{\Delta t}^q$
- $\widetilde{V}_{h,\Delta t}^{p,q}$ subspace vanishing at boundary



Discretized mixed formulation

Find $(u_{h_1,\Delta t_1}, \lambda_{h_2,\Delta t_2}) \in \widetilde{V}^{1,1}_{h_1,\Delta t_1} \times V^{0,0,+}_{h_2,\Delta t_2}$ such that

$$\begin{cases} (a)\ \langle \mathcal{S} u_{h_1,\Delta t_1}, v_{h_1,\Delta t_1} \rangle - \langle \lambda_{h_2,\Delta t_2}, v_{h_1,\Delta t_1} \rangle = \langle h, v_{h_1,\Delta t_1} \rangle \\ (b)\ \langle u_{h_1,\Delta t_1}, \mu_{h_2,\Delta t_2} - \lambda_{h_2,\Delta t_2} \rangle \geq 0, \end{cases}$$

for all $(v_{h_1,\Delta t_1}, \mu_{h_2,\Delta t_2}) \in \widetilde{V}^{1,1}_{h_1,\Delta t_1} \times V^{0,0,+}_{h_2,\Delta t_2}$.

Theorem: Both the continuous and discretized variational inequalities admit unique solutions.

Theorem: Uzawa algorithm converges for discrete problem.

Discretized mixed formulation

Find $(u_{h_1,\Delta t_1}, \lambda_{h_2,\Delta t_2}) \in \widetilde{V}^{1,1}_{h_1,\Delta t_1} \times V^{0,0,+}_{h_2,\Delta t_2}$ such that

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for all $(v_{h_1,\Delta t_1}, \mu_{h_2,\Delta t_2}) \in \widetilde{V}^{1,1}_{h_1,\Delta t_1} \times V^{0,0,+}_{h_2,\Delta t_2}$.

Error analysis based on:

Theorem (space–time $\inf - \sup$)

Let $C > 0$ sufficiently small, and $\frac{\max\{h_1,\Delta t_1\}}{\min\{h_2,\Delta t_2\}} < C$. Then there exists $\alpha > 0$ such that for all $\lambda_{\Delta t_2,h_2}$:

$$\sup_{v_{\Delta t_1,h_1}} \frac{\langle v_{\Delta t_1,h_1}, \lambda_{\Delta t_2,h_2} \rangle}{\| v_{\Delta t_1,h_1} \|_{0,\frac{1}{2},\sigma,*}} \geq \alpha \| \lambda_{\Delta t_2,h_2} \|_{0,-\frac{1}{2},\sigma} .$$

Discretized mixed formulation

Find $(u_{h_1,\Delta t_1}, \lambda_{h_2,\Delta t_2}) \in \widetilde{V}^{1,1}_{h_1,\Delta t_1} \times V^{0,0,+}_{h_2,\Delta t_2}$ such that

$$\begin{cases} (a)\ \langle \mathcal{S} u_{h_1,\Delta t_1}, v_{h_1,\Delta t_1}\rangle - \langle \lambda_{h_2,\Delta t_2}, v_{h_1,\Delta t_1}\rangle = \langle h, v_{h_1,\Delta t_1}\rangle \\ (b)\ \langle u_{h_1,\Delta t_1}, \mu_{h_2,\Delta t_2} - \lambda_{h_2,\Delta t_2}\rangle \geq 0, \end{cases}$$

for all $(v_{h_1,\Delta t_1}, \mu_{h_2,\Delta t_2}) \in \widetilde{V}^{1,1}_{h_1,\Delta t_1} \times V^{0,0,+}_{h_2,\Delta t_2}$.

The inf-sup condition implies a priori estimates similar to Brezzi – Hager – Raviart '78:

Theorem (a priori error estimate)

$$\|\lambda - \lambda_{\Delta t_2,h_2}\|_{0,-\frac{1}{2},\sigma} \lesssim \inf_{\tilde{\lambda}_{\Delta t_2,h_2}} \|\lambda - \tilde{\lambda}_{\Delta t_2,h_2}\|_{0,-\frac{1}{2},\sigma} + (\Delta t_1)^{-\frac{1}{2}} \|u - u_{\Delta t_1,h_1}\|_{-\frac{1}{2}}$$

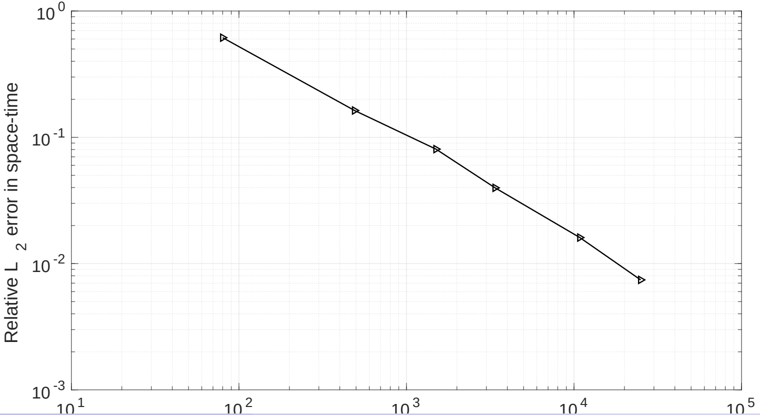
$$\|u - u_{\Delta t_1,h_1}\|_{-\frac{1}{2},\frac{1}{2},\sigma,*} \lesssim_\sigma \inf_{v_{\Delta t_1,h_1}} \|u - v_{\Delta t_1,h_1}\|_{\frac{1}{2},\frac{1}{2},\sigma,*} + \inf_{\tilde{\lambda}_{\Delta t_2,h_2}} \Big\{ \|\tilde{\lambda}_{\Delta t_2,h_2} - \lambda\|_{\frac{1}{2},-\frac{1}{2},\sigma} + \|\tilde{\lambda}_{\Delta t_2,h_2} - \lambda_{\Delta t_2,h_2}\|_{\frac{1}{2},-}$$

Experiment 1: Contact $\mathcal{S}u \geq h$ on flat screen

$\Gamma = [-2,2]^2 \times \{0\} \supset G = [-1,1]^2 \times \{0\}$, $0 < t < 5$, $\frac{\Delta t}{\Delta x} = 0.7$

$h = e^{-2t} t^4 \cos(2\pi x)\cos(2\pi y)\chi_{[-0.25,0.25]}(x)\chi_{[-0.25,0.25]}(y)$

relative $L^2([0,T] \times \Gamma)$-error against benchmark with 12800 triangles, $\Delta t = 0.05$

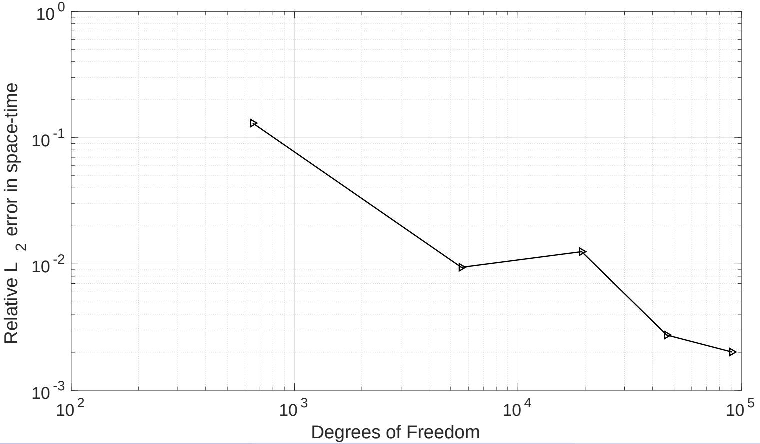


Experiment 2: $\mathcal{S}u \geq h$ with non-flat contact

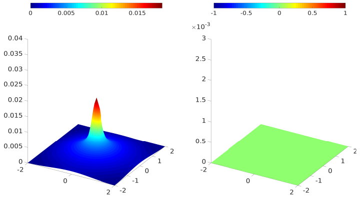
$\Gamma = [-2,2]^3 \supset \Gamma_c = $ 3 faces (top, left, back), $0 < t < 4$, $\frac{\Delta t}{\Delta x} = 0.7$

h as before, on every contact face

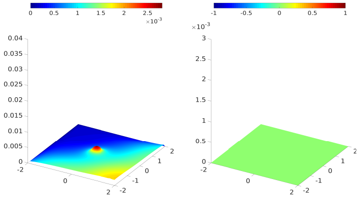
relative $L^2([0,T] \times \Gamma)$-error against benchmark



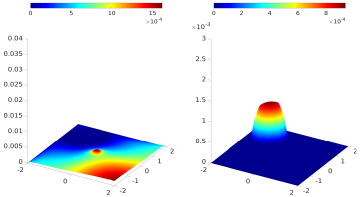
Experiment 2: $\mathcal{S}u \geq h$ with non-flat contact



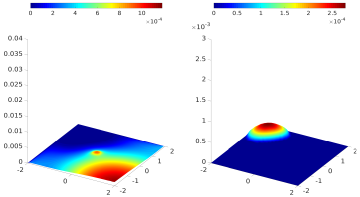
(a) t=3



(b) t=5



(c) t=5.5



(d) t=6

Conclusions

- Boundary elements solve static or dynamic contact problem on Γ
- Barbosa-Hughes stabilization of hp-methods for BEM:
  a priori estimates for Tresca friction: convergence, estimates not optimal due to $S \neq S_{hp}$
  a posteriori error estimates for Tresca and Coulomb
- a priori estimates for dynamic contact in cases where existence of solutions is known
- Recent & current work on time domain BEM:
  a posteriori analysis and adaptivity, not yet contact (with Stephan)
  stabilization of dynamic contact (with Barrenechea)

L. Banz, HG, A. Issaoui, E. P. Stephan, Stabilized mixed hp-BEM for frictional contact problems in linear elasticity, Numer. Math. 135 (2017), 217-263.

HG, F. Meyer, C. Özdemir, E. P. Stephan, Time domain boundary elements for dynamic contact problems, CMAME 333 (2018), 147-175.