Non-uniqueness in plane fluid flows: *an example*

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(joint with Michael Grinfeld[2], Robin J. Knops[3], Marshall Slemrod[4])

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3rd Austrian Calculus of Variations Day
Wien, 23-24 November 2023

arXiv:2301.09122 + ongoing work

Non-uniqueness of weak solutions for Navier-Stokes

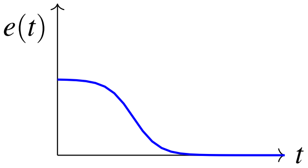

Figure: Smooth energy profile.

Theorem (Buckmaster-Vicol 2019, using convex integration)

There exists $\beta > 0$ *such that for any non-negative smooth function* $e(t) : [0, T] \to \mathbb{R}_{\geq 0}$ *there exists a weak solution* $\mathbf{v} \in C_t^0\left([0, T]; H_x^\beta(\mathbb{T}^3)\right)$ *of the Navier-Stokes equations such that*

$$\int_{\mathbb{T}^3} |\mathbf{v}(x, t)|^2 \, dx = e(t) \qquad \text{for all } t \in [0, T].$$

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Glimm-Lazarev-Chen 2020 propose ***maximal entropy rate criterion***

Aim: Rigorous analysis in toy problem

A toy problem from control theory

Artstein 1983 / Dafermos 2012 consider a dynamical system for $(x(t), y(t)) \in \mathbb{R}^2$ with related properties:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} = \frac{y}{x}, \\ \dot{y} &= -\frac{\partial H}{\partial x} = \frac{y^2 - x^2}{2x^2}, \end{aligned}$$

Hamiltonian $H(x, y) = \frac{x^2+y^2}{2x}$ conserved

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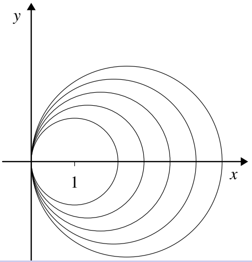
level sets $\{H = c\}$ are circles:

$H(x,y) = c \iff x^2 + y^2 = 2cx \iff (x-c)^2 + y^2 = c^2$



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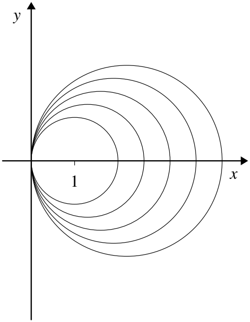

Non-uniqueness occurs when trajectories switch orbits at the common point of intersection.

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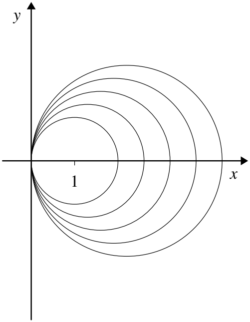

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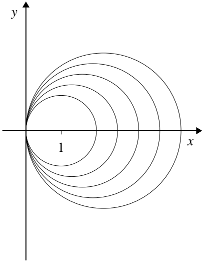
A toy problem from control theory

Artstein 1983 / Dafermos 2012: with control $w = x$

$$\dot{x} = \frac{\partial H}{\partial y} w = y,$$
$$\dot{y} = -\frac{\partial H}{\partial x} w = \frac{y^2 - x^2}{2x},$$



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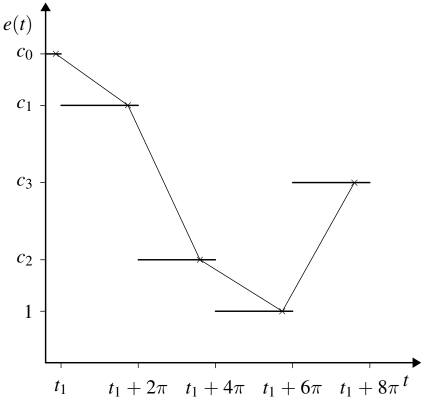
Dafermos also replaces the equations inside the circle of radius 1 by a harmonic oscillator.

Observations

1. **Any** discrete energy profile can be obtained.



Observations

1. **Any** discrete energy profile can be obtained.

2. **Maximal entropy rate criterion** selects unique energy profile. The corresponding solution to the ODE lies on the unit circle.

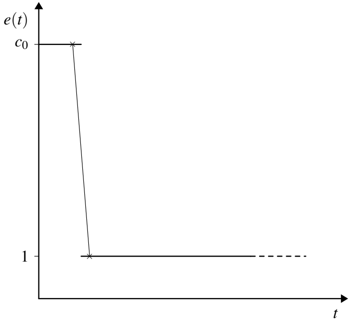


Figure: Energy profile selected by maximal entropy rate criterion.

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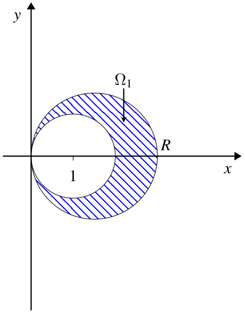

Figure: The region $\Omega = \Omega_1 \cup \Omega_2$.

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continuity equation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0.$$

balance of steady linear momentum with pressure $p(x, y) = \frac{x^2+y^2}{2x} - 1$:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\rho u^2\right) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial p}{\partial x} &= 0, \\ \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}\left(\rho v^2\right) + \frac{\partial p}{\partial y} &= 0, \end{aligned}$$

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boundary conditions:

$$(un_1 + vn_2) = 0, \qquad (x, y) \in \partial\Omega_1 \backslash (0, 0),$$

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4. Singular density $\rho(x)$ leads to **fluid particle aggregation** for the entropy rate admissible trajectories, not for "wild" non-admissible ones.

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5. The velocity field $(u, v) = (\dot{x}, \dot{y})$ is weak solution to the **Navier-Stokes equations**.

viscosity coefficients $\mu,\ \lambda$ given by

$$\begin{aligned} \mu &= 4\cos^2\theta, \\ \lambda &= -4\theta\cot\theta - 4\cos^2\theta. \end{aligned}$$

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5. The velocity field $(u, v) = (\dot{x}, \dot{y})$ is weak solution to the **Navier-Stokes equations**.

6. **Symplectic Euler method** satisfies maximal entropy rate criterion.

Illustration: Symplectic Euler method satisfies maximal entropy rate criterion

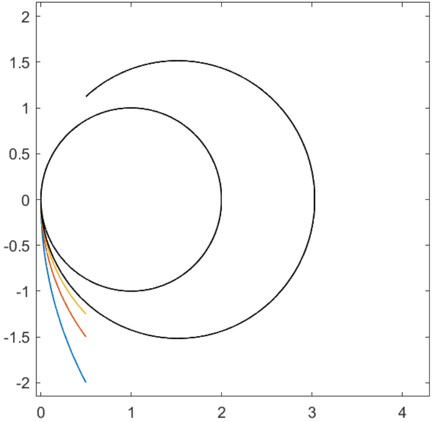

Figure: Numerical trajectories for different initial conditions.

Conclusions

- Singular dynamical system by Artstein / Dafermos gives interesting non-unique planar fluid flows.
- Maximal entropy rate criterion selects unique solution for Lagrangian fluid trajectories.
- Stable numerical approximation using symplectic Euler method.
- Rigorous analysis in this toy problem supports proposals for general fluid equations.

Ongoing work: Artstein / Dafermos - type solutions in elasticity.

arXiv:2301.09122 + ongoing work with Grinfeld, Knops, Slemrod.