THE WILLMORE ENERGY AND THE MAGNITUDE OF EUCLIDEAN DOMAINS

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Abstract. We study the geometric significance of Leinster's notion of magnitude for a compact metric space. For a smooth, compact domain X in an odd-dimensional Euclidean space, we show that the asymptotic expansion of the function $\mathcal{M}_X(R) = \mathrm{Mag}(R\cdot X)$ at $R=\infty$ determines the Willmore energy of the boundary ∂X. This disproves the Leinster-Willerton conjecture for a compact convex body in odd dimensions.

Introduction

The notion of magnitude was introduced by Leinster [8, 9] as an extension of the Euler characteristic to (finite) enriched categories. Magnitude has been shown to unify notions of "size" like the cardinality of a set, the length of an interval or the Euler characteristic of a triangulated manifold, and it even relates to measures of the diversity of a biological system. See [10] for an overview.

Viewing a metric space as a category enriched over $[0,\infty)$, Leinster and Willerton proposed and studied the magnitude of metric spaces [9, 11]: If (X,d) is a finite metric space, a weight function is a function $w : X \to \mathbb{R}$ which satisfies $\sum_{y\in X} \mathrm{e}^{-\mathrm{d}(x,y)} w(y) = 1$ for all $x \in X$. Given a weight function w, we define the magnitude of X as $\mathrm{Mag}(X) := \sum_{x\in X} w(x)$; this definition is independent of the choice of weight function. Beyond finite metric spaces, the magnitude of a compact, positive definite metric space (X,d) was made rigorous by Meckes [12]:

$$\mathrm{Mag}(X) := \sup\{\mathrm{Mag}(\Xi) : \Xi \subset X \quad \text{finite}\}\ .$$

Instead of the magnitude of an individual space (X,d), it proves fruitful to study the magnitude function $\mathcal{M}_X(R) := \mathrm{Mag}(X, R\cdot d)$ for $R>0$.

Compact convex subsets $X \subset \mathbb{R}^n$ provide a key example, surveyed in [10]. Motivated by properties of the Euler characteristic and computer calculations, Leinster and Willerton [11] conjectured a surprising relation to the intrinsic volumes $V_i(X)$, which would shed light on the geometric content of the magnitude function:

$$\mathcal{M}_X(R) = \sum_{k=0}^{n} \frac{1}{k!\omega_k} V_k(X) R^k + o(1), \quad \text{as } R\to\infty. \tag{1}$$

Here, ω_k is the volume of the k-dimensional unit ball. This asymptotic expansion resembles the well-known expansion of the heat trace, with leading terms $V_n(X) = \mathrm{vol}_n(X)$, $V_{n-1}(X) = \mathrm{vol}_{n-1}(\partial X)$ [4]. The expansion coefficients for the heat trace, however, are not proportional to $V_k(X)$ for $k \leq n-2$.

The conjectured behavior (1) was disproved by Barceló and Carbery [1] for the unit ball $B_5 \subset \mathbb{R}^5$. They explicitly computed the rational function $\mathcal{M}_{B_5}$ and observed numerical disagreement of the coefficients of R^k. Their results were extended to balls in odd dimensions in [14].

In spite of this negative result, the authors were able to prove a variant of (1), with modified prefactors, which confirmed the close relation between magnitude and intrinsic volumes [2]: When $n = 2m-1$ is odd and $X \subseteq \mathbb{R}^n$ is a compact domain with smooth boundary, there are coefficients $(c_j(X))_{j\in\mathbb{N}}$ such that

$$\mathcal{M}_X(R) = \sum_{j=0}^{\infty} \frac{c_j(X)}{n!\omega_n} R^{n-j} + O(R^{-\infty}), \quad \text{as } R \to \infty,$$

where

$$c_0(X) = \mathrm{vol}_n(X),\ c_1(X) = m\mathrm{vol}_{n-1}(\partial X),\ c_2(X) = \frac{m^2}{2}\ (n-1)\int_{\partial X} H\,\mathrm{d}S\ .$$

Here, H denotes the mean curvature of ∂X. Each coefficient c_j is an integral over ∂X computable from the second fundamental form of ∂X and its covariant derivatives. For $j = 0, 1, 2$ and X convex, the coefficient c_j is proportional to the intrinsic volume $V_{n-j}(X)$, for $j = 0, 1, 2$. This proves that the Leinster-Willerton conjecture holds for modified universal coefficients up to $O(R^{n-3})$.

The following variant of the Leinster-Willerton conjecture therefore remained plausible. It would confirm the relation between magnitude and intrinsic volumes and, in particular, show that c_n is proportional to the Euler characteristic V_0:

Conjecture 1. *For $n > 0$, there are universal constants $\gamma_{0,n}, \gamma_{1,n}, \ldots, \gamma_{n,n}$ such that for any compact convex subset $X \subseteq \mathbb{R}^n$, $\mathcal{M}_X(R) = \sum_{k=0}^{n} \gamma_{k,n} V_k(X) R^k + o(1)$, as $R \to \infty$.*

In this paper we prove that Conjecture 1 fails in all odd dimensions $n \geq 3$ and find unexpected geometric content in c_3. While the conjecture holds true for the terms of order R^n, R^{n-1} and R^{n-2}, the R^{n-3}-term is not proportional to an intrinsic volume:

Theorem 2. *Assume that $n \geq 3$ is odd and that $X \subseteq \mathbb{R}^n$ is a compact domain with smooth boundary. Then there is a dimensional constant $\lambda_n \neq 0$ such that*

$$c_3(X) = \lambda_n \mathcal{W}(\partial X),$$

where $\mathcal{W}(\partial X) := \int_{\partial X} H^2 \mathrm{d}S$ is the Willmore energy of the boundary of the hypersurface ∂X.

Building on [2], the proof reformulates the magnitude function in terms of an elliptic boundary value problem of order $n+1$ in $\mathbb{R}^n \setminus X$, which is then studied using methods from semiclassical analysis. See Proposition 4 and Equation (5) below.

To see that Theorem 2 disproves Conjecture 1 in the fourth term, we observe that the Willmore energy is not an intrinsic volume: The only intrinsic volume with the same scaling property as the Willmore energy is V_{n-3}. For instance, if $n = 3$ then V_{n-3} is the Euler characteristic while $\int_{\partial X} H^2 \mathrm{d}S$ can be non-zero even when ∂X has vanishing Euler characteristic (e.g. for a torus). In general dimension, for

$a>0$ the solid ellipsoid

$$X_a := \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'|^2 + \frac{|x_n|^2}{a^2} \leq 1 \right\},$$

satisfies that $\mathcal{W}(\partial X_a) \to \infty$ as $a \to 0$. On the other hand, Hausdorff continuity of intrinsic volumes shows that $V_{n-3}(X_a)$ converges to a finite number, namely the $n-3$:rd intrinsic volume of the $n-1$-dimensional unit ball. Therefore Theorem 2 implies the following.

Corollary 3. *Assume that $n \geq 3$ is odd and that $X \subseteq \mathbb{R}^n$ is a compact convex domain with smooth boundary. There are universal constants $\gamma_{n-2,n}, \gamma_{n-1,n}, \gamma_{n,n}$ such that*

$$\mathcal{M}_X(R) = \sum_{k=n-2}^{n} \gamma_{k,n} V_k(X) R^k + O(R^{n-3}), \quad \text{as } R \to \infty.$$

However, there is no constant $\gamma_{n-3,n}$ such that $\mathcal{M}_X(R) = \sum_{k=n-3}^{n} \gamma_{k,n} V_k(X) R^k + O(R^{n-4})$ as $R \to \infty$. In particular, the Leinster-Willerton conjecture fails even with modified universal coefficients.

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Background and notation

We assume that $X \subseteq \mathbb{R}^n$ is a compact domain with C^∞-boundary, where $n = 2m-1$ odd. Denote by $\Omega := \mathbb{R}^n \setminus X$ *the exterior domain*. We use the Sobolev spaces $H^s(\mathbb{R}^n) := (1-\Delta)^{-s/2} L^2(\mathbb{R}^n)$ of exponent $s \geq 0$. Here, the Laplacian Δ is given by $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. The spaces $H^s(X)$ and $H^s(\Omega)$ are defined using restrictions. The Sobolev spaces $H^s(\partial X)$ can be defined using local charts or as $(1-\Delta_{\partial X})^{-s/2} L^2(\partial X)$.

We use ∂_ν to denote the Neumann trace of a function u in Ω. The operator ∂_ν extends to a continuous operator $H^s(\Omega) \to H^{s-3/2}(\partial X)$ for $s > 3/2$. Similarly, $\gamma_0 : H^s(\Omega) \to H^{s-1/2}(\partial X)$ denotes the trace operator defined for $s > 1/2$.

For $R > 0$ we shall need the operators

$$\mathcal{D}_R^j := \begin{cases} \partial_\nu \circ (R^2 - \Delta)^{(j-1)/2}, & \text{when } j \text{ is odd,} \\ \gamma_0 \circ (R^2 - \Delta)^{j/2}, & \text{when } j \text{ is even.} \end{cases}$$

By the trace theorem, $\mathcal{D}_R^j$ is continuous as an operator $\mathcal{D}_R^j : H^s(\Omega) \to H^{s-j-1/2}(\partial X)$ for $s > j + 1/2$.

We recall a key observation from [1], in the reformulation presented in [2]:

Proposition 4. [2, Proposition 9] *Suppose that $h_R \in H^{2m}(\Omega)$ is the unique weak solution to the boundary value problem*

$$\begin{cases} (R^2 - \Delta)^m h_R &= 0 \quad \text{in } \Omega \\ \mathcal{D}_R^j h_R &= \begin{cases} R^j, & j \text{ even} \\ 0, & j \text{ odd.} \end{cases}, \ j = 0, ..., m-1. \end{cases}$$

Then the following identity holds

$$\mathcal{M}_X(R) = \frac{\mathrm{vol}_n(X)}{n!\omega_n} R^n - \frac{1}{n!\omega_n} \sum_{\frac{m}{2} < j \leq m} R^{n-2j} \int_{\partial X} \mathcal{D}_R^{2j-1} h_R \,\mathrm{d}S.$$

The operators $\mathcal{D}_R^j$ define a matrix-valued Dirichlet-Neumann operator $\Lambda(R) : \mathcal{H}_+ \to \mathcal{H}_-$ in the Hilbert space

$$\mathcal{H} := \underbrace{\bigoplus_{j=0}^{m-1} H^{2m-j-1/2}(\partial X)}_{\mathcal{H}_+} \oplus \underbrace{\bigoplus_{j=m}^{n} H^{2m-j-1/2}(\partial X)}_{\mathcal{H}_-}$$

as follows: $\Lambda(R)(u_j)_{j=0}^{m-1} := (\mathcal{D}_R^j u)_{j=m}^n$, where $u \in H^{2m}(\Omega)$ is the unique weak solution to

$$\begin{cases} (R^2 - \Delta)^m u &= 0 \quad \text{in } \Omega \\ \mathcal{D}_R^j u &= u_j \ \ , \ j = 0, ..., m-1. \end{cases} \tag{2}$$

The operator $\Lambda(R)$ is a parameter-dependent pseudodifferential operator on ∂X. The parameter R enters like an additional co-variable, which allows us to compute the asymptotics of $\mathcal{M}_X$ from Proposition 4. For the convenience of the reader we recall the salient features of the parameter-dependent pseudodifferential calculus, see for instance [5, 6, 13] for further details. We restrict to parameters $R \in \mathbb{R}_+ = (0, \infty)$.

Definition 5. A parameter-dependent pseudodifferential operator A of order s on $\mathbb{R}^n$ is an operator on the Schwartz space of the form

$$Af(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi, R) \mathrm{e}^{i(x-y)\xi} f(y) \mathrm{d}y \mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{3}$$

where the full symbol a admits a polyhomogeneous expansion of order s in (ξ, R). That is, for $k \in \mathbb{N}$ there are functions $a_{s-k} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+)$ with

$$a_{s-k}(x, t\xi, tR) = t^{s-k} a_{s-k}(x, \xi, R), \quad \text{for } t \geq 1, \, \|(\xi, R)\| \geq 1,$$

and a can be written as an asymptotic sum

$$a \sim \sum_{k=0}^{\infty} a_{s-k}.$$

We call a_s the principal symbol of A. If $a_s(x, \xi, R)$ is invertible for every $(x, \xi, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, we say that A is elliptic with parameter.

Definition 5 on $\mathbb{R}^n$ extends by standard techniques, using coordinate charts, to define a pseudodifferential operator and its full symbol on a compact manifold, see for instance [2, 5, 6, 13]. The use of the parameter-dependent calculus is crucial to the work [2] and the computations in this paper, including formulas for the symbol of a product of two pseudodifferential operators and the parametrix construction. In particular, if A is elliptic with parameter of order s on a compact manifold, it has a parametrix with parameter B of order $-s$. The full symbol expansion

$b \sim \sum_{j=0}^{\infty} b_{-s-j}$ can be explicitly computed: The principal symbol is given by $b_{-s}(x,\xi,R) = a_s(x,\xi,R)^{-1}$, and for $j>0$ the following inductive formula holds,

$$b_{-s-j}(x,\xi,R) = -a_s(x,\xi,R)^{-1} \sum_{\substack{k+l+|\alpha|=j,\\ l<j}} \frac{i^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a_{s-k}(x,\xi,R) \partial_x^\alpha b_{-s-l}(x,\xi,R). \tag{4}$$

The computation proving Equation (4) follows from [13, Section 5.5].

For $R \to \infty$ the parameter-dependent calculus further allows to compute expectation values of the form $\int_M A(1)\, \mathrm{d}x$ in terms of the symbol:

Lemma 6. *Suppose that $A : C^\infty(M) \to C^\infty(M)$ is a parameter-dependent pseudodifferential operator of order s acting on a compact manifold M equipped with a volume density. Then there is an asymptotic expansion*

$$\int_M A(1)\, \mathrm{d}x = \sum_{k=0}^{\infty} a_k R^{s-k} + O(R^{-\infty}),$$

where the coefficients a_k are computed as follows: Expand the full symbol of A into terms homogeneous in (ξ, R) as $\sigma_A(x,\xi,R) \sim \sum_{k=0}^{\infty} \sigma_{s-k}(A)(x,\xi,R)$ and set

$$a_k := \int_M \sigma_{s-k}(A)(x,0,1)\, \mathrm{d}x.$$

For the proof of Lemma 6 we refer the reader to [2, Lemma 20] or [3, Lemma 2.24], but let us outline the main idea. The claimed asymptotics of Lemma 6 is coordinate invariant because $\int_M A(1)\, \mathrm{d}x$ is coordinate invariant. It therefore suffices to compute the asymptotics for an operator A on $\mathbb{R}^n$ as in Equation (3), assuming a is compactly supported in the x-variable. In this case, $A(1) = a(x,0,R)$, so that for $R \geq 1$

$$\int_{\mathbb{R}^n} A(1)\, \mathrm{d}x = \int_{\mathbb{R}^n} a(x,0,R) \mathrm{d}x = \sum_{k=0}^{\infty} \int_M \sigma_{s-k}(A)(x,0,R)\, \mathrm{d}x + O(R^{-\infty}) =$$
$$= \sum_{k=0}^{\infty} \int_M \sigma_{s-k}(A)(x,0,1)\, \mathrm{d}x R^{s-k} + O(R^{-\infty}) = \sum_{k=0}^{\infty} a_k R^{s-k} + O(R^{-\infty}).$$

The reader should note that the integrands $a_{s-k}(x,0,R) = a_{s-k}(x,0,1)R^{m-k}$ are well defined because each a_{s-k} is homogeneous in (ξ, R), and not only in ξ.

From Proposition 4 and Lemma 6 we deduce a formula for the expansion coefficients c_k:

$$c_k(X) := -\sum_{\frac{m}{2}<j\leq m} \sum_{0\leq l<m/2} \int_{\partial X} \sigma_{2j-2l-k}(\Lambda_{2j-1,2l})(x,0,1)\, \mathrm{d}S, \tag{5}$$

for $k>0$ where $\Lambda = (\Lambda_{j+m,l})_{j,l=0}^{m-1}$ and $\sigma_{2j-2l-k}(\Lambda_{2j-1,2l})$ the homogeneous part of order $2j-2l-k$ in its symbol (with parameter). See [2, Proposition 20].

The full symbol of the parameter-dependent operator Λ can be computed by adapting standard techniques in semiclassical analysis [6]. The operator Λ is first computed using boundary layer potentials. To define these, we consider the function

$$K(R;z) := \frac{\kappa_n}{R} \mathrm{e}^{-R|z|}, \quad z \in \mathbb{R}^n.$$

The constant $\kappa_n > 0$ is chosen such that

$$(R^2 - \Delta)^m K = \delta_0$$

in the sense of distributions on $\mathbb{R}^n$. For $l = 0, \dots, n$, we define the functions

$$K_l(R; x, y) := (-1)^l \mathcal{D}^{n-l}_{R,y} K(R; x-y), \quad x \in \mathbb{R}^n,\ y \in \partial X.$$

Here $\mathcal{D}^l_{R,y}$ denotes $\mathcal{D}^l_R$ acting in the y-variable. We also consider the distributions

$$K_{j,k}(R; x, y) := \mathcal{D}^j_{R,x} K_k(R; x, y), \quad x \in \partial X.$$

Each $K_{j,k}$ defines a parameter-dependent pseudodifferential operator $A_{j,k}(R) : C^\infty(\partial X) \to C^\infty(\partial X)$,

$$A_{j,k}(R) f(x) := \int_{\partial X} K_{j,k}(R; x, y) f(y) \mathrm{d}S(y), \quad x \in \partial X.$$

The integral defining $A_{j,k}(R)$ is understood in the sense of an exterior limit. These operators combine into a $2m \times 2m$-matrix of operators $\mathbb{A} := (A_{j,l})^n_{j,l=0} : \mathcal{H} \to \mathcal{H}$. It decomposes into matrix blocks

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_{++} & \mathbb{A}_{+-} \\ \mathbb{A}_{-+} & \mathbb{A}_{--} \end{pmatrix} : \begin{matrix} \mathcal{H}_+ \\ \oplus \\ \mathcal{H}_- \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_+ \\ \oplus \\ \mathcal{H}_- \end{matrix},$$

with $\mathbb{A}_{pq} : \mathcal{H}_q \to \mathcal{H}_p$ for $p, q \in \{+, -\}$. By integrating by parts as in [2, Proposition 12], one can show that if u solves Equation (2) then

$$u_+ = \mathbb{A}_{++} u_+ + \mathbb{A}_{+-} u_-,$$

where $u_+ := (u_j)^{m-1}_{j=0}$ and $u_- := (u_{m+j})^{m-1}_{j=0}$. Therefore, $(1 - \mathbb{A}_{++}) u_+ = \mathbb{A}_{+-} u_-$ and we can express the Dirichlet-Neumann operator Λ in terms of layer potentials as

$$\Lambda = \mathbb{B}(1 - \mathbb{A}_{++}). \tag{6}$$

Here $\mathbb{B} = (B_{j+m,l})^{m-1}_{j,l=0}$ denotes a parametrix (with parameter) of $\mathbb{A}_{+-} = (A_{j,l+m})^{m-1}_{j,l=0}$. See more in the proof of [2, Theorem 18].

The proof of Theorem 2 uses Equation (6) to compute components of the symbol of the Dirichlet-Neumann operator Λ. The formula for c_3 then follows from (5).

Proof of Theorem 2

To prove Theorem 2 we note that we by Equation (5) only need to compute the third term $\sigma_{2j-2l-3}(\Lambda_{2j-1,2l})$ in the polyhomogeneous expansion

$$\sigma(\Lambda_{2j-1,2l})(x, \xi, R) \sim \sum_{k=0}^{\infty} \sigma_{2j-2l-1-k}(\Lambda_{2j-1,2l})(x, \xi, R),$$

in the range $\frac{m}{2} < j \leq m$, $0 \leq l < m/2$. In fact, we only need to compute the evaluation $\sigma_{2j-2l-3}(\Lambda_{2j-1,2l})(x, 0, 1)$. Recall that we are using the parameter-dependent calculus, so that each $\sigma_{2j-2l-1-k}(\Lambda_{2j-1,2l})(x, \xi, R)$ is homogeneous of degree $-2j-2l-1-k$ in (ξ, R).

For the convenience of the reader, we change to the notation $(x', \xi', R) \in T^*\partial X \times \mathbb{R}_+$ for coordinates and cotangent variables on the boundary ∂X, as used in [2]. For an integer $k \in \mathbb{Z}$, we use the notation

$$\begin{aligned} \sigma_k(\mathbb{A}_{++}) &:= (\sigma_{j-l+k}(A_{j,l}))^{m-1}_{j,l=0}, \\ \sigma_k(\mathbb{A}_{+-}) &:= (\sigma_{j-l+k-m}(A_{j,l+m}))^{m-1}_{j,l=0} \quad \text{and} \\ \sigma_k(\mathbb{B}) &:= (\sigma_{j+m-l+k}(B_{j+m,l}))^{m-1}_{j,l=0}. \end{aligned}$$

Here we write $\sigma_{j-l+k}(A_{j,l})$ for the degree $j-l+k$ part of $a_{j,l}$ written as a symbol depending on the variable $(x',\xi',R) \in T^*\partial X \times \mathbb{R}_+$. The symbols $\sigma_k(\mathbb{A}_{++})$, $\sigma_k(\mathbb{A}_{+-})$ and $\sigma_k(\mathbb{B})$ relate to the (parameter-dependent) Douglis-Nirenberg calculus naturally appearing in the boundary reduction of boundary value problems [2, 5]. The reader should note the difference with the expressions appearing just after [2, Proposition 37] in that they are for symbols in the variables (x',y',ξ',R). The process of going between these two symbol expressions is one of the difficulties in the computation ahead.

The reader can note that $\sigma_0(\mathbb{A}_{++})$, $\sigma_0(\mathbb{A}_{+-})$ and $\sigma_0(\mathbb{B})$ are the matrices of principal symbols of $\mathbb{A}_{++}$, $\mathbb{A}_{+-}$ and $\mathbb{B}$, respectively. In particular,

$$\sigma_0(\mathbb{B}) = \sigma_0(\mathbb{A}_{+-})^{-1}.$$

It follows from [2, Theorem 12] that $\sigma_0(\mathbb{B})$ does not depend on $x' \in \partial X$. Define the symbol

$$\mathbb{D} = (\delta_{j,k}(R^2+|\xi|^2)^{j/2})_{j,k=0}^n.$$

By the computational result [2, Theorem 12], there are constant $m\times m$-matrices C_0, C_1, C_2, C_3 such that

$$\begin{aligned} \sigma_0(\mathbb{A}_{++}) &= \mathbb{D}C_0\mathbb{D}^{-1}, \quad \sigma_0(\mathbb{A}_{+-}) = \mathbb{D}C_1\mathbb{D}^{-1}, \\ \sigma_{-1}(\mathbb{A}_{++}) &= H\mathbb{D}C_2\mathbb{D}^{-1}, \quad \sigma_{-1}(\mathbb{A}_{+-}) = H\mathbb{D}C_3\mathbb{D}^{-1}, \quad \text{and} \\ \sigma_0(\mathbb{B}) &= \mathbb{D}C_1^{-1}\mathbb{D}^{-1}, \end{aligned}$$

where H denotes the mean curvature of ∂X and we in each identity embed $m\times m$-matrices in a suitable fashion into $2m\times 2m$-matrices.

From [2, Lemma 22, part a] and the x'-independence of $\sigma_0(\mathbb{B})$ we can from Equation (4) deduce that

$$\sigma_{-1}(\mathbb{B}) = -\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{+-})\sigma_0(\mathbb{B}) = H\mathbb{D}C_1^{-1}C_3C_1^{-1}\mathbb{D}^{-1},$$

as well as

$$\begin{aligned} \sigma_{-2}(\mathbb{B}) = -\sigma_0(\mathbb{B})\Bigg(&\sigma_{-2}(\mathbb{A}_{+-}) + \sum_{j=1}^{n-1}\partial_{\xi_j}\sigma_0(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\partial_{x_j}\sigma_{-1}(\mathbb{A}_{+-}) - \\ &-\sigma_{-1}(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{+-})\Bigg)\sigma_0(\mathbb{B}). \end{aligned}$$

Using [2, Lemma 22, part b], we write

$$\begin{aligned}\sigma_{-2}(\Lambda) =&\sigma_{-2}(\mathbb{B})(1-\sigma_0(\mathbb{A}_{++}))-\sigma_{-1}(\mathbb{B})\sigma_{-1}(\mathbb{A}_{++})-\sigma_0(\mathbb{B})\sigma_{-2}(\mathbb{A}_{++})+\\ &+i\sum_{j=1}^{n-1}\partial_{\xi_j}\sigma_{-1}(\mathbb{B})\partial_{x_j}\sigma_{-1}(\mathbb{A}_{++})=\\ =&-\sigma_0(\mathbb{B})\Big(\sigma_{-2}(\mathbb{A}_{+-})+\sum_{j=1}^{n-1}\partial_{\xi_j}\sigma_0(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\partial_{x_j}\sigma_{-1}(\mathbb{A}_{+-})-\\ &-\sigma_{-1}(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{+-})\Big)\sigma_0(\mathbb{B})(1-\sigma_0(\mathbb{A}_{++}))+\\ &+\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{++}))-\sigma_0(\mathbb{B})\sigma_{-2}(\mathbb{A}_{++})-\\ &-i\sum_{j=1}^{n-1}\partial_{\xi_j}\left(\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\right)\partial_{x_j}\sigma_{-1}(\mathbb{A}_{++})\end{aligned}$$

Since all σ_0-occurences only depend on $R^2+|\xi|^2$, all its ξ-derivatives will vanish at $\xi=0$, and therefore,

$$\begin{aligned}&\sigma_{-2}(\Lambda)(x',0,R)=\\ &=\Big[-\sigma_0(\mathbb{B})\Big(\sigma_{-2}(\mathbb{A}_{+-})-\sigma_{-1}(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{+-})\Big)\sigma_0(\mathbb{B})(1-\sigma_0(\mathbb{A}_{++}))+\\ &\quad+\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{+-})\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{A}_{++})-\sigma_0(\mathbb{B})\sigma_{-2}(\mathbb{A}_{++})\Big]_{\xi'=0}=\\ &=\Big[-\mathbb{D}C_1^{-1}\mathbb{D}^{-1}\left(\sigma_{-2}(\mathbb{A}_{+-})\mathbb{D}C_1^{-1}(1-C_0)\mathbb{D}^{-1}+\sigma_{-2}(\mathbb{A}_{++})\right)+\\ &\quad+H^2\mathbb{D}C_1^{-1}C_3C_1^{-1}C_3C_1^{-1}(1-C_0)\mathbb{D}^{-1}+H^2\mathbb{D}C_1^{-1}C_3C_1^{-1}C_2\mathbb{D}^{-1}\Big]_{\xi'=0}.\end{aligned}$$

Assume for now that $\sigma_{-2}(\mathbb{A}_{+-})(x',0,R)=\sigma_{-2}(\mathbb{A}_{++})(x',0,R)=0$. Then this computation shows that indeed, there are universal constants $(d_{j+m,l})_{j,l=0}^{m-1}$ (independent of X) such that for $\frac{m}{2}<j\leq m$ and $0\leq l<m/2$,

$$\sigma_{2j-2l-2}(\Lambda_{2j-1,2l})(x,0,1)=d_{2j-1,2l}H(x)^2.$$

In particular, we have shown that for a dimensional constant λ_n, we have that $c_3(X)=\lambda_n\int_{\partial X}H^2\mathrm{d}S$. It follows from [14] that $\lambda_n\neq 0$ for $n\geq 3$ odd.

It remains to show that $\sigma_{-2}(\mathbb{A}_{+-})(x',0,R)=\sigma_{-2}(\mathbb{A}_{++})(x',0,R)=0$. Note that we do not claim that $\sigma_{-2}(\mathbb{A}_{+-})=\sigma_{-2}(\mathbb{A}_{++})=0$ just that when restricting to $\xi'=0$ the symbols vanish. This last step in the proof relies on the technically involved computations in [2, Appendix A.2] and the process of going from "two-variable symbols" $\tilde{a}(x,y,\xi,R)$ to "one-variable symbols" $a(x,\xi,R)$, see [7, Theorem 7.13]. We pick local coordinates at a point on ∂X. We can assume that this point is $0\in\mathbb{R}^n$ and that the coordinates are of the form $(x',S(x'))$, where x' belongs to some neighborhood of $0\in\mathbb{R}^{n-1}$ and S is a scalar function with $S(0)=0$ and

$\nabla S(0)=0$. We can express a_{jk} as

$$\begin{aligned}
a_{jk}(x',y',\xi',R) =& b_{0,m-p-q}(R^2+|\xi'|^2, S(x')-S(y')),\\
&\text{when } j=2p, k=n-2q\\
a_{jk}(x',y',\xi',R) =& b_{1,m-p-q}(R^2+|\xi'|^2, S(x')-S(y'))+\\
&(\xi'\cdot\nabla S(x'))b_{0,m-p-q}(R^2+|\xi'|^2, S(x')-S(y')),\\
&\text{when } j=2p+1, k=n-2q
\end{aligned}$$

$$\begin{aligned}
a_{jk}(x',y',\xi',R) =& b_{1,m-p-q}(R^2+|\xi'|^2, S(x')-S(y'))+\\
&(\xi'\cdot\nabla S(y'))b_{0,m-p-q}(R^2+|\xi'|^2, S(x')-S(y')),\\
&\text{when } j=2p, k=n-2q-1\\
a_{jk}(x',y',\xi',R) =& b_{2,m-p-q}(R^2+|\xi'|^2, S(x')-S(y'))+\\
&((\xi'\cdot\nabla S(y'))+(\xi'\cdot\nabla S(x')))b_{1,m-p-q}(R^2+|\xi'|^2, S(x')-S(y'))+\\
&(\xi'\cdot\nabla S(x'))(\xi'\cdot\nabla S(y'))b_{0,m-p-q}(R^2+|\xi'|^2, S(x')-S(y')),\\
&\text{when } j=2p+1, k=n-2q-1,
\end{aligned}$$

where

$$b_{r,N}(u,z)=\begin{cases}(-i\partial_z)^r(u-\partial_z^2)^{-N}\delta_{z=0}, & N\leq 0,\\ \\ (-i\partial_z)^r\sum_{k=0}^{N-1}\tilde{c}_{k,r,N}\frac{|z|^k\mathrm{e}^{-|z|\sqrt{u}}}{u^{N-(k+1)/2}}, & N>0,\end{cases}$$

for some coefficients $\tilde{c}_{k,r,N}$.

We need to verify that $\sigma_{j-k-2}(A_{j,k})(x',0,R)=0$ for any j and k. The symbol $\sigma_{j-k-2}(A_{j,k})$ in $x'=0$ is by [7, Theorem 7.13] given by the terms of order $j-k-2$ in the expression

$$a_{jk}(0,0,\xi',R)-i\sum_{l=1}^{n-1}\frac{\partial^2 a_{jk}}{\partial\xi_l\partial y_l}(0,0,\xi',R)-\frac{1}{2}\sum_{l,s=1}^{n-1}\frac{\partial^4 a_{jk}}{\partial\xi_l\partial\xi_s\partial y_l\partial y_s}(0,0,\xi',R)$$

Recall that $S(0)=0$ and $\nabla S(0)=0$ so there are several terms vanishing when setting $x'=0$. Indeed, no term of order $j-k-2$ in $a_{jk}(0,0,\xi',R)$ is non-zero. All non-zero terms of order $j-k-2$ in $\sum_{l=1}^{n-1}\frac{\partial^2 a_{jk}}{\partial\xi_l\partial y_l}(0,0,\xi',R)$ are odd functions under the reflection $\xi'\mapsto-\xi'$, so they vanish when restricting to $\xi'=0$. Similar computations show that terms of order $j-k-2$ in $\frac{1}{2}\sum_{l,s=1}^{n-1}\frac{\partial^4 a_{jk}}{\partial\xi_l\partial\xi_s\partial y_l\partial y_s}$ all contains a factor of ξ_l or $\xi_l\xi_s$ so they vanish when restricting to $\xi'=0$.

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