The Fundamental Matrix of the System of Linear Elastodynamics in Hexagonal Media. Solution to the Problem of Conical Refraction

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Abstract. An explicit integral representation by single definite integrals of the fundamental matrix (Green’s tensor) of the time-dependent system of hexagonal elastic media is derived. Thereby the problem of internal conical refraction in such media is solved.

0. Introduction and notations

Hexagonal or transversely isotropic media are characterized by the property of rotational symmetry with respect to an axis. In this paper, we extend R. G. Payton’s seminal work in that area by providing qualitative and quantitative information on the fundamental matrix \( E \) (= “free space Green’s tensor”) of the elastodynamic system

\[
P(\partial) = P(\partial_t, \nabla) = I_3 \partial_t^2 + A(\nabla)
\]

for hexagonal crystals. On the one hand, we determine the singular support of \( E \) (Prop. 4), thereby solving the open problem of conical refraction; on the other hand, we represent \( E \) by simple definite integrals (Prop. 5).

Up to now, representations of \( E \) were given by loop integrals over curves defined as intersections of the slowness surface \( \det P(1, \xi) = 0 \) with the planes \( \Pi_{(t, x)} = \{ \xi \in \mathbb{R}^3; t + x\xi = 0 \} \), see [5, (5.4), p. 45], [31, (6.7), (6.8), p. 3318], [38, (A), p. 327], or formula (8) in Prop. 3 below. Though these loop integrals running over curves of genus 3 (in general) yield some geometric insight into the structure of \( E \), they are harder to evaluate numerically than the simple integrals in Prop. 5. Also, the specialization to particular cases is not as straightforward, cf. e.g. [38, Section 4, p. 336].
In the recent article [7], R. Burridge and J. Qian take the loop integral approach for the system of crystal optics. Let us mention, incidentally, that the “static term” (see [7, (5.11), p. 78]) was calculated for the magnetic field in [38, p. 335], cf. also [49, p. 415]. We also remark that, for the electric field, the static term $E_1$ can be represented more simply than in [7], namely $E_1 = -\frac{Y(t)}{4\pi \sqrt{\det \sigma}} \nabla \nabla^T \frac{1}{\sqrt{\sigma^{-1} x^T \sigma^{-1} x}}$.

Our representation of $E$ starts from a variant of the Herglotz–Petrovsky–Leray formula for the fundamental matrix of a system with a strictly hyperbolic determinant, which formula we call the Herglotz–Gårding formula (see §2, Prop. 1). In our analysis of the qualitative properties of $E$ in §3, we first give necessary and sufficient conditions for the hyperbolicity of the operator $P(\partial)$ in terms of the elastic constants (see Prop. 2), and then adjust the formula of Prop. 1 to the system of hexagonal elastodynamics, which is not strictly hyperbolic (Prop. 3).

Next we determine in Prop. 4 the singular support of $E$, i.e. the set where $E$ is not smooth. Its intersection with $t = 1$, the so-called wave-front surface is, essentially, the dual of the slowness surface. However, “conical” (i.e. singular) points on the slowness surface can result in (internal) conical refraction. Mathematically, this amounts to additional plane or conical lids contained in the wave-front surface. (These lids are given by the supports of the corresponding localized operators.) For scalar operators in three space dimensions, these lids are always present due to the foundational results of M. F. Atiyah, R. Bott, and L. Gårding (cf. e.g. [2, Th. 7.7, p. 175]), whereas for systems, these lids can disappear. (This phenomenon is connected with the form of the adjoint matrix of $P$; it is most easily understood when considering a diagonal $2 \times 2$ system $P = (P_1 \, P_2)$: Points $\xi_0$ which belong to the intersection of the slowness surfaces $P_1(1, \xi) = 0$, $P_2(1, \xi) = 0$ and are regular on both of them are singular on the slowness surface of $P(\partial)$, i.e. of the determinant operator $\det P(\partial) = P_1(\partial)P_2(\partial)$, without leading to conical refraction for the system $P(\partial)$. In fact, $P(\partial)$ has the fundamental matrix $(E_i \, 0)$ if $E_i$ is the fundamental solution of $P_i(\partial)$, $i = 1, 2$.) For a general account on the history and mathematics of conical refraction, cf. [17] and [35].

We shall show that conical refraction does not occur in the propagation of elastic waves in general hexagonal media (see Prop. 4 and the Remark 1 thereupon). This qualitative analysis is related to the historical development as follows: In 1954 and 1957, J. de Klerk and M. J. P. Musgrave predicted conical refraction for transverse elastic waves in cubic and in tetragonal crystals as a result of their investigation of the conical points on the slowness surface. A proof of the existence of conical refraction was given for cubic crystals by R. Burridge [5] in 1967, and for tetragonal crystals by P. A. Barry and M. J. P. Musgrave in 1978 [3]. In 1983, R. G. Payton [41, pp. 66, 67] conjectured that conical refraction did not occur in transversely isotropic elastic solids. However, in 1992 [42], he proved the existence of conical refraction in special hexagonal media fulfilling $c_{11} = c_{33} = c_{44}$. The peculiarity of the cases $c_{11} = c_{44}$, $c_{33} = c_{44}$, and $c_{13} + c_{44} = 0$, respectively, was also recognized and investigated by P. Chadwick and A. L. Shuvalov [11] in 1997.

As mentioned above, we shall show that no wave-front lids exist in general trans-
versely isotropic solids, and we shall characterize precisely the limit cases in which such lids (i.e. conical refraction) appear.

The definite integrals we obtain in Prop. 5 for the fundamental matrix are, in general, complete Abelian integrals of genus 3. The calculation of the genus of the complex algebraic curve connected with the integral representation (cf. Rem. 1 to Prop. 5) clarifies the nature of these integrals. They reduce to integrals of genus 0 (i.e. algebraic functions) iff

\[ c_{13} + c_{44} = 0 \quad \text{or} \quad (c_{13} + c_{44})^2 = (c_{11} - c_{44})(c_{33} - c_{44}), \]

for which cases we deduce in § 5 the known results of R. G. Payton ([41], cf. also [6], [38]). Furthermore, a reduction of the general representation formula to elliptic integrals (i.e. genus 1) occurs if \( c_{11} = c_{44} \), a particular case of which was studied in [42].

In § 6, the values of the fundamental matrix in the \( x_1, x_2 \)-plane are expressed by elliptic integrals. Different representation formulas for this case were already deduced in 1967 by N. Cameron and G. Eason [8]. Finally, we calculate the values of \( E \) on the \( x_3 \)-axis (see Prop. 6) and obtain the results of R. G. Payton [40], [41].

We make constant use of L. Schwartz’ theory of distributions ([46]), since we agree with L. Gårding that, before the invention of distribution theory, “the analysis of singularities of solutions to partial differential equations was a painful matter, especially for non-elliptic ones. A real understanding first became possible with the framework of Schwartz’s distributions ...” ([15, p. 32]).

Finally, let us establish some notations. As usual, Euclidean space is written as \( \mathbb{R}^n \) and \( S^{n-1} \) denotes the unit sphere in \( \mathbb{R}^n \), i.e., \( S^{n-1} := \{ x \in \mathbb{R}^n : |x| = 1 \} \).

In general, the variables in \( \mathbb{R}^n \) are written as \( (t, x) = (t, x_1, \ldots, x_{n-1}) \) or \( (\tau, \xi) = (\tau, \xi_1, \ldots, \xi_{n-1}) \). The Heaviside function is denoted by \( Y \), the \( l \times l \) unit matrix is written as \( I_l \), and \( A^{ad} \) is the adjoint matrix of the \( l \times l \) matrix \( A \), i.e., \( A \cdot A^{ad} = (\det A) \cdot I_l \).

We consider differential operators with constant coefficients only and use as differentiation symbols

\[ \partial_t := \frac{\partial}{\partial t}, \quad \nabla := (\partial_1, \ldots, \partial_{n-1}) := \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}} \right), \quad \partial := (\partial_t, \nabla), \quad \Delta_l := \partial_1^2 + \cdots + \partial_l^2. \]

We employ the standard notations for the distribution spaces \( \mathcal{D}', \mathcal{S}' \), the duals of the spaces \( \mathcal{D}, \mathcal{S} \) of “test functions” and of “rapidly decreasing functions”, respectively, cf. [13], [19], [26], [28], [45], [46], and we use angle brackets, e.g. \( \langle \phi, T \rangle \), to indicate the evaluation of the distribution \( T \) on the test function \( \phi \). By supp \( T \), sing supp \( T \), sing supp \( A T \), we denote the support, the singular support, and the analytic singular support of \( T \), respectively, i.e. the complements of the sets, where \( T \) vanishes, is infinitely differentiable, or is real-analytic, respectively, cf. [26, 2.2 and 8.4], [13, 1.4.1, 8.6.1]. We use the Fourier transform \( \mathcal{F} \) in the form \( (\mathcal{F}\phi)(x) := \int e^{-i\xi x} \phi(\xi) \, d\xi, \; \phi \in \mathcal{S} \), this being extended to \( \mathcal{S}' \) by continuity. Herein and also elsewhere, the Euclidean inner product \( (x, \xi) \mapsto x\xi \) is simply expressed by juxtaposition.
In several places, we use the abbreviations \( \rho^2 := \xi_1^2 + \xi_2^2 \), \( x' := (x_1, x_2) \), \( H_+ := \{(t, x) \in \mathbb{R}^4; t \geq 0\} \) and \( \Pi(t, x) := \{\xi \in \mathbb{R}^3; t + x\xi = 0\} \).

For the convenience of the reader, the constants \( a_1, \ldots, a_7 \) and the wave operators \( W_1, \ldots, W_5 \), which appear scattered in the text, are collected in an appendix.

1. The equations of linear elastodynamics in transversely isotropic media

Let us recall the equations governing the displacement in transversely isotropic media, cf. [41, (1.1.6), (1.3.2), pp. 1, 3], [35, (4.5.1b), p. 57, p. 94]. In anisotropic media, the laws of Newton and of Hooke yield

\[
(\rho I_3 \partial_t^2 + A(\nabla)) u = \rho f,
\]

where \( \rho \) denotes the (constant) density of mass, \( u \) and \( f \) represent the vectors of displacement and of exterior forces (per unit mass), respectively, and \( A(\nabla) \) is a matrix of second-order linear differential operators determined by the tensor \( (c_{pqrs}) \) of the elastic constants. The dimension of the linear space of tensors \( (c_{pqrs}) \) fulfilling the appropriate symmetry relations equals 21 in general, but reduces to 5 in the case of transversely isotropic media. Following [4, (6.7), (6.10), pp. 572, 573] and [38, 1.2.3, p. 319], we abbreviate

\[
a_1 = c_{1111} = c_{11}, \quad a_2 = c_{3333} = c_{33}, \quad a_3 = c_{1133} + c_{2323} = c_{13} + c_{44}, \quad a_4 = \frac{1}{2}(c_{1111} - c_{1122}) = \frac{1}{2}(c_{11} - c_{12}), \quad a_5 = c_{2323} = c_{44},
\]

\((c_{ij}\) being the “contracted index notation”, cf. [41, (1.3.1), p. 3]), and then obtain

\[
A(\nabla) = -\begin{pmatrix}
  a_1 \partial_1^2 + a_4 \partial_2^2 + a_5 \partial_3^2 & (a_1 - a_4) \partial_1 \partial_2 & a_3 \partial_1 \partial_3 \\
  (a_1 - a_4) \partial_1 \partial_2 & a_4 \partial_2^2 + a_5 \partial_3^2 & a_3 \partial_2 \partial_3 \\
  a_3 \partial_1 \partial_3 & a_3 \partial_2 \partial_3 & a_5 (\partial_1^2 + \partial_2^2) + a_2 \partial_3^2
\end{pmatrix}.
\]

Without restriction of generality, we set the mass density \( \rho \) equal to 1. The solution \( u \) of (1) in unbounded space is given as a convolution integral of the fundamental matrix \( E \) with \( f \). Here and in the following, \( E \) denotes the uniquely determined matrix of distributions with support in the half-space \( H_+ := \{(t, x) \in \mathbb{R}^4; t \geq 0\} \) fulfilling the matrix equation

\[
( I_3 \partial_t^2 + A(\nabla)) E = I_3 \delta,
\]

where \( \delta \) denotes the Dirac measure at the origin. Hence the \( i \)-th column of \( E \) corresponds to the exterior force vector \( e_i \delta \) with \( I_3 = (e_1 | e_2 | e_3) \). (Note that there is no generally accepted term for the concept of fundamental matrix. In physics, it is often called “Green’s tensor” (cf. [41]) or “free space Green’s tensor”, in the mathematical literature “noyau élémentaire à droite” (cf. [46, p. 140]), “fundamental solution to the right” (cf. [25, 3.8, p. 94]), “Green’s matrix” (cf. [20, III.1, p. 106]), “fundamental solution” (cf. [16, p. 215]).)
Similarly as in [12], [9], [35], [41], we shall illustrate our calculations by two specific transversely isotropic materials, namely cobalt and titanium boride (TiB$_2$), which are characterized by the following elastic constants (in gigapascal):

- Cobalt: $a_1 = 307$, $a_2 = 358$, $a_3 = 178.5$, $a_4 = 71$, $a_5 = 75.5$
- TiB$_2$: $a_1 = 690$, $a_2 = 440$, $a_3 = 570$, $a_4 = 140$, $a_5 = 250$

see [41, Table 1, p. 3], [9, Table 4, p. 37].

2. The Herglotz–Gårding formula for the fundamental matrix of hyperbolic systems

For the explicit representation of the fundamental matrix of a homogeneous system $P(\partial)$ with strictly hyperbolic determinant, we shall deduce a modification of the Herglotz–Petrovsky–Leray formula (see [1, pp. 173–177], [27, (12.6.10), p. 129]), which modification we call the Herglotz–Gårding formula, since it is derived from the formula in [17, Thm. 2, p. 375] for the fundamental solution of a homogeneous, strictly hyperbolic scalar operator.

$P(\partial)$ denotes an $l \times l$ matrix of constant coefficient differential operators acting on $\mathbb{R}^n$ and homogeneous of degree $m$ and we always set $Q := \det P$. (In §1, $P(\partial)$ corresponds to $I_3 \partial^2_t + A(\nabla)$. ) $P(\partial)$ is assumed to be hyperbolic with respect to $t$, i.e.

(i) $Q(1,0) \neq 0$, and

(ii) the polynomial $\tau \mapsto Q(\tau, \xi)$ has only real roots for each $\xi \in \mathbb{R}^{n-1}$,

cf. [1, p. 129], [29, pp. 89, 90], [27, Thm. 12.4.3, p. 113]. If these roots are pairwise different for each $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$, then $Q$ is called strictly hyperbolic, cf. [1, Def. 3.8, p. 129], [27, Def. 12.4.11, p. 118]. (Note that the strict hyperbolicity of $P$, i.e. the hyperbolicity of $P + P_1$ for all matrices $P_1$ of differential operators of lower order, is implied by, but is not equivalent to, the strict hyperbolicity of $Q$, cf. [30, p. 787], [16, p. 221].)

**Proposition 1.** Let $P(\tau, \xi) = P(\tau, \xi_1, \ldots, \xi_{n-1})$ be a real $l \times l$ matrix of polynomials which are homogeneous of degree $m$ and suppose that $Q(\partial) = \det P(\partial)$ is strictly hyperbolic with respect to $t$ and that $Q(\tau, \xi)$ does not contain $\tau$ as a factor. Define the measure $T \in D'(\mathbb{R}^n \setminus \{0\})^{l \times l}$ by

$$T := P^{\text{ad}}(\tau, \xi) \delta(Q(\tau, \xi)) \text{sign}((\partial_\tau Q)(\tau, \xi)).$$

Furthermore, set $s^\lambda_+ := Y(s) s^\lambda \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_+)$ for $\Re \lambda > -1$ and let $s^{n-m-1}_+$ be the finite part evaluated at $n - m - 1$ of the meromorphic extension to the whole complex plane of the holomorphic function

$$\{ \lambda \in \mathbb{C}; \Re \lambda > -1 \} \longrightarrow \mathcal{S}'(\mathbb{R}) : \lambda \mapsto s^\lambda_+,$$
Then the uniquely determined fundamental matrix $E$ of $P(\partial)$ with support in $H_+$ fulfills

$$(4) \quad E(t, x) = -2(2\pi)^{1-n} Y(t) \int_{\mathbb{R}^{n-1}} T(1, \xi) \text{Re}[i^{m+1} F s_+^{n-m-1}] (t + x \xi) \, d\xi + Y(t) S(t, x),$$

where $S = 0$ if $n$ is even or $m < n$, and is otherwise an $l \times l$ matrix of homogeneous polynomials of degree $m - n$.

Remarks. 1) Due to the homogeneity of $T$, the restriction of $T$ to the hyperplane $\tau = 1$ is well-defined; furthermore, the integral $\int_{\mathbb{R}^{n-1}} \cdots d\xi$ in (4) has to be understood in the distributional sense, i.e. for $\phi \in D(\mathbb{R}_{t,x}^n)$ with $\text{supp} \phi \subset H_+$, we have

$$\langle \phi, E \rangle = -2(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} T(1, \xi) \psi(\xi) \, d\xi + \langle \phi, S \rangle \in \mathcal{C}^{1 \times l}$$

where $T(1, \xi) \psi(\xi)$ is an $l \times l$ matrix of integrable measures. Note also that the multiplication with $Y(t)$ in formula (4) is well-defined, since the support of the following distribution intersects the hyperplane $t = 0$ in the origin $x = 0$ only, and since a homogeneous distribution of degree $m - n$ can uniquely be continued from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}^n$ (for $m \geq 1$).

2) Formula (4), which we call the Herglotz–Gårding formula, generalizes earlier formulas such as [23, (4)–(13), pp. 609, 610], [24, (7.58), p. 192], [1, pp. 176, 177], [17, Thm. 2, p. 375], [38, Thm., p. 324], [51, Prop. 1, p. 309].

Proof. a) By the hyperbolicity of $P$, the matrices $P(\tau \pm i\epsilon, \xi)$ are invertible for $\epsilon > 0$ and $(\tau, \xi) \in \mathbb{R}^n$, and the entries of the inverse matrices $P(\tau \pm i\epsilon, \xi)^{-1}$ grow at most polynomially when $\epsilon \searrow 0$. Hence the two limits $\lim_{\epsilon \searrow 0} P(\tau \pm i\epsilon, \xi)^{-1}$ exist in $\mathcal{D}'(\mathbb{R}^n)^{l \times l}$ (cf. [1, p. 121]) and yield homogeneous distributions of degree $-m$.

Since

$$\mathcal{F}^{-1} \left( \lim_{\epsilon \searrow 0} P(\tau \pm i\epsilon, \xi)^{-1} \right)$$

are the two fundamental matrices of $P(-i\partial_t, -i\nabla)$ with support in $\mp H_+$ respectively (cf. [27, (12.5.3), p. 120]), we obtain

$$E = Y(t)i^{1-m}2\pi \mathcal{F}^{-1} T$$

with $T = \frac{1}{2\pi i} \lim_{\epsilon \searrow 0} (P(\tau - i\epsilon, \xi)^{-1} - P(\tau + i\epsilon, \xi)^{-1}) \in \mathcal{S}'(\mathbb{R}^n)^{l \times l}$. 

Next, due to the strict hyperbolicity of $Q(\partial) = \det P(\partial)$, Sokhotsky’s formula

$$\lim_{\epsilon \searrow 0} \frac{1}{x \pm i\epsilon} = \text{vp} \frac{1}{x} \mp i\pi \delta \quad \text{in} \, \mathcal{D}'(\mathbb{R}^1)$$
implies (outside the origin)
\[
T = \frac{1}{2\pi i} P^{ad}(\tau, \xi) \lim_{\epsilon \to 0} (Q(\tau - i\epsilon, \xi)^{-1} - Q(\tau + i\epsilon, \xi)^{-1})
\]
\[
= P^{ad}(\tau, \xi) \delta(Q(\tau, \xi)) \text{sign}(\partial_{\tau} Q)(\tau, \xi),
\]

cf. [38, p. 322]. Note that the last expression is defined in $D'(\mathbb{R}^n \backslash \{0\})$ in the usual way:
\[
\langle \phi, \delta(Q(\tau, \xi)) \text{sign}(\partial_{\tau} Q)(\tau, \xi) \rangle = \frac{d}{ds} \left( \int_{Q(\tau, \xi) < s} \phi(\tau, \xi) \text{sign}(\partial_{\tau} Q)(\tau, \xi) \, d\tau d\xi \right) \bigg|_{s=0},
\]

cf. [13, p. 82].

b) Let us abbreviate $(\tau, \xi)$ by $\eta$. We will represent the inverse Fourier transform of the homogeneous distribution $T \in S'(\mathbb{R}^n)$ as a surface integral by introducing polar coordinates. $T$ has the restriction $T|_{S^{n-1}} \in D'(S^{n-1})$ and $T$ coincides outside the origin with $T|_{S^{n-1}} \cdot |\eta|^{-m}$, which is defined in the same way as $s_{n-m-1}^+ \lambda$ (see above) as the finite part of the meromorphic continuation of $\lambda \mapsto T|_{S^{n-1}} \cdot |\lambda|^n$, $\text{Re}\lambda > -n$. Because of $T = (-1)^{m+1}T$ we also have $\hat{U} = (-1)^{m+1}U$ if
\[
U := T - T|_{S^{n-1}} \cdot |\eta|^{-m}.
\]

On the other hand, due to $\text{supp}\, U \subset \{0\}$ and due to the homogeneity of $U$, we have $\hat{U} = (-1)^{m+n}U$ and therefore $U$ vanishes for even $n$ or if $m < n$.

By [14, Lemme 6.2, p. 406] or [39, Cor. 2.6.3],
\[
\langle \phi, F^{-1}(T|_{S^{n-1}} \cdot |\eta|^{-m}) \rangle = (2\pi)^{-n} \langle \phi(t, x), F s_{n-m-1}^+ (-t\tau - x\xi), T|_{S^{n-1}} \rangle
\]
and hence
\[
E(t, x) = 2\pi Y(t) \text{Re}(i^m F^{-1}T)
\]
\[
= -(2\pi)^{-n} Y(t) \text{Re}[i^{m+1} F s_{n-m-1}^+ (t\tau + x\xi), T|_{S^{n-1}}] + Y(t) S(t, x),
\]

wherein taking the real part is justified by the reality of $P$ and of $U$. Moreover, $S = 2\pi \text{Re}(i^{1-m} F^{-1}U)$ is a matrix of polynomials.

c) Finally, we make use of the “gnomonian projection”, i.e. the diffeomorphism
\[
\{-1, 1\} \times \mathbb{R}^{n-1} \to \{\eta \in S^{n-1}; \eta_1 \neq 0\} : (\tau, \xi) \mapsto \frac{(\tau, \xi)}{|(\tau, \xi)|}.
\]
For a function $f \in L^1(S^{n-1})$, we have
\[
\int_{S^{n-1}} f(\eta) \sigma(\eta) = \int_{\mathbb{R}^{n-1}} \left[ f \left( \frac{(1, \xi)}{\sqrt{1 + |\xi|^2}} \right) + f \left( \frac{(-1, \xi)}{\sqrt{1 + |\xi|^2}} \right) \right] \frac{d\xi}{(1 + |\xi|^2)^n/2},
\]
if $\sigma$ denotes the surface measure on $S^{n-1}$. In particular, if $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$ is homogeneous of degree $-n$ and even, then

$$\int_{S^{n-1}} f(\eta) \sigma(\eta) = 2 \int_{\mathbb{R}^{n-1}} f(1, \xi) \, d\xi.$$  

Similarly, if $\mu$ is a Radon measure on $\mathbb{R}^n \setminus \{0\}$ which fulfills

$$\mu(\{ \eta \in \mathbb{R}^n; \eta_1 = 0 \}) = 0,$$

and which, as an element of $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, is even and homogeneous of degree $-n$, then

$$\int_{S^{n-1}} \mu|_{S^{n-1}} = 2 \int_{\mathbb{R}^{n-1}} \mu|_{\{1\} \times \mathbb{R}^{n-1}}.$$  

By the homogeneity of $\mu$, the restrictions $\mu|_{S^{n-1}}$ and $\mu|_{\{1\} \times \mathbb{R}^{n-1}}$ are well-defined Radon measures.

In our case,

$$\mu(\eta) = \mu(\tau, \xi) = \langle \phi(t, x), \text{Re}[i^{m+1} F_{s^+}^{n-m-1}](t \tau + x\xi) \rangle \cdot T$$

is even and the hyperplane $\{ \eta \in \mathbb{R}^n; \eta_1 = 0 \}$ is a null-set with respect to $\mu$, since $Q(\tau, \xi)$ does not contain $\tau$ as a factor. Furthermore, $\mu$ is homogeneous in $\mathbb{R}^n \setminus \{0\}$ except for $m \geq n$ and odd $n$. In this last case, $\mu(\eta)$ is associated homogeneous (see [19, p. 83]) and the gnomonian projection changes just the polynomial term $S(t, x)$.

Hence, we conclude that

$$\langle \phi, E \rangle = -2(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} T(1, \xi) \psi(\xi) \, d\xi + \langle \phi, S \rangle,$$

where

$$\psi(\xi) = \langle \phi(t, x), \text{Re}[i^{m+1} F_{s^+}^{n-m-1}](t + x\xi) \rangle$$

is a $C^\infty$ function,

$$T(1, \xi) = P^{ad}(1, \xi) \delta(Q(1, \xi)) \text{sign}( (\partial_\tau Q)(1, \xi))$$

is a matrix of Radon measures, $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp} \phi \subset H_+$, and $T(1, \xi) \psi(\xi)$ is a matrix of integrable measures (comp. [28, p. 345]).

**Remark.** If the dimension $n$ equals 4 and the degree $m$ of homogeneity is 2, then we have

$$\text{Re}[i^{m+1} F_{s^+}^{n-m-1}] = \text{Re}(-iF_{s^+}) = \pi \delta',$$

and hence the fundamental matrix $E$ of $P(\partial)$ with support in $H_+$ is given by

$$E(t, x) = -\frac{Y(t)}{4\pi^2} \partial_t \int_{\mathbb{R}^3} P^{ad}(1, \xi) \delta(Q(1, \xi)) \text{sign}( (\partial_\tau Q)(1, \xi)) \cdot \delta(t + x\xi) \, d\xi.$$
3. The fundamental matrix of hexagonal elastodynamics: Qualitative aspects

Let us specialize now Prop. 1 and formula (5) to the system of hexagonal elastodynamics, i.e. to

\[ P(\partial) = I_3 \partial_1^2 + A(\nabla) \]

with \(A(\nabla)\) as in formula (3). As observed already by Christoffel in 1877 [41, p. 7], the determinant \(Q = \det P\) splits (cf. [38, p. 320]):

\[ Q = W_1 \cdot R, \quad \text{wherein } W_1(\partial) = \partial_1^2 - a_4 \Delta_2 - a_5 \partial_2^2, \quad a_6 := \frac{1}{2}(a_1 a_2 + a_2^2 - a_3^2) \text{ and } \]

\[ R(\partial) = (\partial_1^2, \Delta_2, \partial_2^2) B \begin{pmatrix} \partial_1^2 \\ \Delta_2 \\ \partial_2^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -\frac{1}{2}(a_1 + a_5) & -\frac{1}{2}(a_2 + a_5) \\ -\frac{1}{2}(a_1 + a_5) & a_1 a_5 & a_6 \\ -\frac{1}{2}(a_2 + a_5) & a_6 & a_2 a_5 \end{pmatrix}. \]

Let us first state necessary and sufficient conditions for the hyperbolicity of \(P\).

**Proposition 2.** The hexagonal elastodynamic system (1), (3) is hyperbolic with respect to \(t\) if and only if the elastic constants fulfill

\[ a_1 \geq 0, \ a_2 \geq 0, \ a_4 \geq 0, \ a_5 \geq 0, \ and \ a_5 + \sqrt{a_1 a_2} \geq |a_3|. \]

**Proof.** Clearly \(P\) is hyperbolic if and only if \(W_1\) and \(R\) are hyperbolic. \(W_1\) is hyperbolic iff \(a_4 \geq 0\) and \(a_5 \geq 0\). On the other hand, the hyperbolicity of \(R\) is characterized by the following conditions on \(B = (b_{ij})\), see [50, Prop. 1, p. 141]:

\[ b_{22} \geq 0, \ b_{33} \geq 0, \ b_{12} \leq -\sqrt{b_{11} b_{22}}, \ b_{13} \leq -\sqrt{b_{11} b_{33}}, \]

\[ b_{23} \geq -\sqrt{b_{22} b_{33}}, \ B_{23}^{ad} \geq -\sqrt{B_{22}^{ad} B_{33}^{ad}}. \]

This means

\[ a_1 a_5 \geq 0, \ a_2 a_5 \geq 0, \ -\frac{a_1 + a_5}{2} \leq -\sqrt{a_1 a_5}, \ -\frac{a_2 + a_5}{2} \leq -\sqrt{a_1 a_5}, \]

\[ a_6 \geq -\sqrt{a_1 a_2 a_5^2} a_3, \ a_3^2 \leq \frac{(a_1 - a_5)(a_2 - a_5)}{4} \geq -\frac{|a_1 - a_5| |a_2 - a_5|}{4}. \]

The first four conditions yield \(a_1 \geq 0, \ a_2 \geq 0, \ a_5 \geq 0\), and the last one is always satisfied. The fifth condition is equivalent to \(a_5 + \sqrt{a_1 a_2} \geq |a_3|\).

**Remarks.** 1) The fourth-order operator \(R\) is strictly hyperbolic iff the six inequalities \(b_{22} \geq 0\) etc. in the proof of Prop. 2 hold in the strict sense (see [50, p. 142]). Hence \(R\) is strictly hyperbolic if and only if

\[ a_1 > 0, \ a_2 > 0, \ a_5 > 0, \ a_1 \neq a_5, \ a_2 \neq a_5, \]

\[ |a_3| < a_5 + \sqrt{a_1 a_2} \quad \text{and} \quad |a_3| \neq 0 \quad \text{or} \quad (a_1 - a_5)(a_2 - a_5) < 0. \]
Similarly, \( W_1 \) is strictly hyperbolic iff \( a_4 > 0 \) and \( a_5 > 0 \).

We point out that, in contrast, the product operator \( Q = W_1 \cdot R \) of order six is never strictly hyperbolic, since the slowness surfaces \( W_1(1, \xi) = 0 \) and \( R(1, \xi) = 0 \) meet in \( \xi_1 = \xi_2 = 0, \xi_3 = \pm \frac{1}{\sqrt{a_5}} \). Moreover, these two slowness surfaces can meet along circles around the \( \xi_3 \)-axis, cf. Figs. 2,3,4, left sides.

2) In the following, we shall suppose—in addition to the hyperbolicity of \( P \)—that the slowness surface \( Q(1, \xi) = 0 \) does not contain points at infinity, i.e. that \( Q(0, \xi) \neq 0 \) for all \( \xi \in \mathbb{R}^3 \setminus \{0\} \). This means that all the inequalities in Prop. 2 are strict. The same assumption is made in [41, (1.55), p. 11] “so that zero propagation speeds are ruled out” (cf. [41, p. 5]). In [41] and in [33] it is shown that these strict inequalities (i.e. (7) below) are equivalent to the “strong ellipticity” of the tensor \( (c_{ijkl}) \). We shall prove in the next proposition that, under that hypothesis, formula (4) in Prop. 1 remains valid if the distribution \( T \) therein is properly interpreted.

**Proposition 3.** Let \( P(\tau, \xi) = \tau^2 I_3 + A(\xi) \) be the system of hexagonal elastodynamics as given by formula (3) and suppose that \( P \) is hyperbolic and that the propagation speed is positive, i.e. that

\[
(7) \quad a_1 > 0, a_2 > 0, a_4 > 0, a_5 > 0, \text{ and } |a_3| < a_5 + \sqrt{a_1 a_2}.
\]

Set \( Q(\tau, \xi) = \det P(\tau, \xi) = \prod_{j=1}^3 (\tau^2 - f_j(\xi)) \). Then the unique fundamental matrix \( E \) of \( P(\partial) \) with support in \( H_+ \) is given by

\[
(8) \quad E = -\frac{Y(t)}{4\pi^2} \partial_t \int_{\mathbb{R}^3} P^{\text{adi}}(1, \xi) S(1, \xi) \delta(t + x\xi) \, d\xi,
\]

where

(i) \( S(\tau, \xi) \in \mathcal{D}'(\mathbb{R}^4) \) is homogeneous of degree \(-6\);

(ii) outside the set where \( Q = \partial_\tau Q = 0, S \) is a Radon measure, and, more precisely,

\[
S = \delta(Q) \cdot \text{sign}(\partial_\tau Q) = \frac{\delta(R)}{W_1} \text{sign}(\partial_\tau R) + \frac{\delta(W_1)}{R} \text{sign}(\tau)
\]

\[
= \text{sign} \tau \cdot \sum_{i=1}^3 \frac{\delta(\tau^2 - f_i(\xi))}{\prod_{j \neq i}(f_i(\xi) - f_j(\xi))};
\]

(iii) in \( \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}) \), we have \( S = \int_{\Sigma_2} \delta''(\tau^2 - g_\lambda(\xi)) \, d\omega(\lambda), \) i.e.

\[
\langle \phi, S \rangle = \frac{1}{8} \int_{\mathbb{R}^3} \int_{\Sigma_2} \frac{(\tau \cdot \nabla)^2 \phi(\sqrt{g_\lambda(\xi)}, \xi) + (\nabla \cdot \tau)^2 \phi(\sqrt{-g_\lambda(\xi)}, \xi)}{\sqrt{g_\lambda(\xi)}} \, d\omega(\lambda) \, d\xi,
\]

where \( g_\lambda(\xi) = \sum_{j=1}^3 \lambda_j f_j(\xi), \ \phi \in \mathcal{D}(\mathbb{R}^4 \setminus \{0\}) \) and

\[\Sigma_2 = \{ \lambda \in \mathbb{R}^3; \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_1 + \lambda_2 + \lambda_3 = 1 \}, \ d\omega(\lambda) = d\lambda_1 d\lambda_2.\]
Proof. a) The first part of the proof of Prop. 1 holds true for hyperbolic systems in general, i.e.,

\[ E = Y(t)i^{1-m}2\pi F^{-1}T \] with \( T = P^{\text{ad}}S \) and

\[ S := \frac{1}{2\pi i} \lim_{\epsilon \to 0} [Q(\tau - i\epsilon, \xi)^{-1} - Q(\tau + i\epsilon, \xi)^{-1}] . \]

The properties (i) and (ii) of \( S \) follow as in the strictly hyperbolic case. Note that

\[ \delta(F \cdot G) = \frac{\delta(F)}{|G|} + \frac{\delta(G)}{|F|} \]

if \( F, G \) are submersive \( C^\infty \) functions (i.e. with non-vanishing gradients) and without common zeros. (In the corresponding formula in [19, p. 236], the absolute value signs are missing.)

Furthermore, if \( \Omega \subset \mathbb{R}^n \) is open, \( f : \Omega \to (0, \infty) \) is \( C^\infty \), and \( \phi \in \mathcal{D}(\mathbb{R} \times \Omega) \), then

\[ \langle \phi, \delta(\tau^2 - f(\xi)) \rangle = \frac{d}{ds} \bigg|_{s=0} \int_{\tau^2 - f(\xi) < s} \phi(\tau, \xi) \, d\tau d\xi \]

\[ = \int_{\Omega} \frac{\phi(\sqrt{f(\xi)}, \xi) + \phi(-\sqrt{f(\xi)}, \xi)}{2 \sqrt{f(\xi)}} \, d\xi, \]

cf. [13, p. 82].

b) In order to derive a representation of \( S \) which is generally valid, we make use of Feynman’s first formula, cf. see [45, p. 72], [44, 18, 3.3.4.3, p. 590]:

\[ \prod_{j=1}^3 a_j^{-1} = 2 \int_{\Sigma_2} \frac{d\omega(\lambda)}{(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3)^3}, \quad a_1 > 0, a_2 > 0, a_3 > 0. \]

By analytic continuation (cf. [37, Proof of Prop. 1, p. 307]), this implies

\[ Q(\tau \pm i\epsilon, \xi)^{-1} = \prod_{j=1}^3 ((\tau \pm i\epsilon)^2 - f_j(\xi))^{-1} = 2 \int_{\Sigma_2} ((\tau \pm i\epsilon)^2 - g_\lambda(\xi))^{-3} d\omega(\lambda). \]

Since \( S = 0 \) for \( \tau = 0, \xi \neq 0 \), we infer that

\[ S = \frac{1}{2\pi i} \lim_{\epsilon \to 0} [Q(\tau - i\epsilon, \xi)^{-1} - Q(\tau + i\epsilon, \xi)^{-1}] \]

\[ = \frac{2}{2\pi i} \lim_{\epsilon \to 0} \int_{\Sigma_2} \left[ ((\tau - i\epsilon)^2 - g_\lambda(\xi))^{-3} - ((\tau + i\epsilon)^2 - g_\lambda(\xi))^{-3} \right] d\omega(\lambda) \]

\[ = \frac{1}{8\pi i} \lim_{\epsilon \to 0} \int_{\Sigma_2} \left( \frac{1}{\tau} \partial_\tau \right)^2 \left[ ((\tau - i\epsilon)^2 - g_\lambda(\xi))^{-1} - ((\tau + i\epsilon)^2 - g_\lambda(\xi))^{-1} \right] d\omega(\lambda). \]
Eventually, Sokhotsky’s formula ([38, p. 322]) yields
\[
S = \frac{1}{4} \int_{\Sigma_2} \left( \frac{1}{\tau} \partial_\tau \right)^2 \delta(\tau^2 - g_\lambda(\xi)) \, d\omega(\lambda) = \int_{\Sigma_2} \delta''(\tau^2 - g_\lambda(\xi)) \, d\omega(\lambda) \text{ in } \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}).
\]

Applied to test functions \( \phi \in \mathcal{D}(\mathbb{R}^4 \setminus \{0\}) \), the first equation furnishes the formula representing \( S \) in (iii).

c) The application of the inverse Fourier transformation yields, as in the proof of Prop. 1,
\[
E = -\frac{Y(t)}{8\pi^2} \left( \delta'(t\tau + x\xi), T|_{\mathbb{R}^4} \right).
\]

Finally, the gnomonian projection can be applied as in part c) of the proof of Prop. 1, since \( S \) vanishes in a neighborhood of the set \( \{(\tau,\xi) \in \mathbb{S}^3; \tau = 0\} \) due to the assumption of positive propagation speed.

Remark. Obviously, in the same way as in Prop. 3, the Herglotz–Gårding formula can be generalized to more general non-strictly hyperbolic operators by using an appropriate definition of the distribution \( T \) in formula (4). More precisely, let \( P(\tau, \xi) \) be a real \( l \times l \) matrix of polynomials which are homogeneous of degree \( m \) and suppose that \( P(\partial) \) is hyperbolic with respect to \( t \), that \( P \) is even in \( \tau \), and that \( \det P(0, \xi) \neq 0 \) for all \( \xi \in \mathbb{R}^{n-1} \setminus \{0\} \). Then formula (4) holds for the fundamental matrix \( E \) of \( P(\partial) \) if \( T \) is defined in \( \mathcal{D}'(\mathbb{R}^n \setminus \{0\})^{l \times l} \) by
\[
T = (-1)^{k-1} P^{ad}(\tau, \xi) \cdot \int_{\Sigma_{k-1}} \delta^{(k-1)}(\tau^2 - g_\lambda(\xi)) \, d\omega(\lambda)
\]
with \( k = \frac{lm}{2} \), \( g_\lambda(\xi) = \sum_{j=1}^{k} \lambda_j f_j(\xi) \), \( \det P(\tau, \xi) = \prod_{j=1}^{k} (\tau^2 - f_j(\xi)) \).

Let us next describe the support and the singular support of \( E \). We consider first the hyperbolic operator \( Q = \det P \) and its fundamental solution \( F \) with support in \( H_+ \). By the general theory of hyperbolic operators ([1], [2], [27]), the component \( \Gamma \) of \( \{\eta \in \mathbb{R}^4; Q(\eta) \neq 0\} \) containing \( (1,0,0,0) \) is convex and
\[
\text{supp } F \subset K := \Gamma^* := \{(t, x) \in \mathbb{R}^4; \forall (\tau, \xi) \in \Gamma : t\tau + x\xi \geq 0\}.
\]
(The cone \( K \) is the convex hull of the support of \( E \) and is called “propagation cone”.)

Furthermore, by [2, Th. 7.7 (d), p. 175], the (analytic) singular support of \( F \) is given by
\[
\text{sing supp } F = \text{sing supp}_A F = \bigcup_{\eta \in \mathbb{R}^4 \setminus \{0\}} \Gamma(Q_\eta)^*;
\]
here \( Q_\eta \) denotes the localization of \( Q \) in the direction \( \eta \) (see [1, 3.36, p. 135]), and \( \Gamma(Q_\eta) \) is the component of \( \{\lambda \in \mathbb{R}^4; Q_\eta(\lambda) \neq 0\} \) containing \( (1,0,0,0) \).
For regular points $\eta$ on the cone $\Xi := \{ \eta \in \mathbb{R}^4; Q(\eta) = 0 \}$ (i.e. for points which satisfy $\nabla Q(\eta) \neq 0$), $Q_\eta$ is an operator of first order and $\Gamma(Q_\eta)^*$ is a half-line on the “dual cone”

\begin{equation}
\Xi^* := \{ \nabla Q(\eta); Q(\eta) = 0 \}
\end{equation}

of $\Xi$. Hence $\text{sing supp } F$ contains $K \cap \Xi^*$.

Geometrically, $\Xi^*$ is the union of all normals to $\Xi$ and we obtain an equation for $\Xi^*$ by observing that $(t, x) \in \Xi^*$ iff the plane $tr + x \xi = 0$ is tangent to the cone $\Xi$, i.e., iff the discriminant of $Q(1, \xi, 0, -(t + \xi_1|x'|)/x_3)$ with respect to $\xi_1$ vanishes. In accordance with Plücker’s formula, $\Xi^*$ is the zero set of a homogeneous polynomial of degree 12, cf. [50, p. 146]. In the following two figures, the intersections with the planes $\xi_2 = 0$ and $x_2 = 0$, respectively, of the “slowness surface” $\{ \xi \in \mathbb{R}^3; Q(1, \xi) = 0 \}$ and of the “wave-front surface” $\{ x \in \mathbb{R}^3; (1, x) \in \Xi^T \}$ are depicted (comp. [43, Fig. 1, p. 309], [9, Fig. 2, p. 433]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{slowness_surface_wave_front_surface_cobalt}
\caption{Slowness surface and wave-front surface for cobalt}
\end{figure}

Let us suppose that $W_1$ and $R$ are strictly hyperbolic and that $\Xi^* \neq \{0\}$, i.e. $Q = W_1 \cdot R$ does not contain a square factor. (This can happen only if $R$ factors into two aligned quadrics, i.e. if $a_3 = 0$ or $a_3^2 = (a_1 - a_5)(a_2 - a_5)$, and these cases are treated in detail in 5.1 and 5.2 below.) Then $Q = 0$ contains as singular points the lines $\xi_1 = \xi_2 = 0$, $\xi_3 = \pm \frac{r}{\sqrt{a_5}}$, and possibly, depending on the values of the elastic constants, circular cones given by $W_1 = R = 0$, $\xi_1^2 + \xi_2^2 \neq 0$ (see Remark 1 to Prop. 2). The corresponding localized operator $Q_\eta(\partial)$ is a multiple of $(\partial_1 \pm \sqrt{a_5} \partial_3)^2$ in the first case and a product $\tilde{W}$ of two first-order operators in the second case. In this second case, $\Gamma(Q_\eta)^*$ is the support of that fundamental solution of $\tilde{W}(\partial)$ which
Figure 2: Slowness surface and wave-front surface for titanium boride

has its support in $H_+$. Therefore, eventually,

$$\text{sing supp } F = (K \cap \Xi^*) \cup \{ t(1,0,0,\pm \sqrt{a_5}); \ t > 0 \} \cup M$$

where $M := \bigcup \Gamma(Q(\tau,\xi))^*.$

The set $M$ represents the so-called “lids of conical refraction”, and they would yield, for $\text{sing supp } F$, four straight bitangent lines in Fig. 2, right part. (They do not belong to the wave-front surface as defined above and they are not depicted in Fig. 2, since $\text{sing supp } E = K \cap \Xi^*$, see Prop. 4.)

The (analytic) singular support of $E$ is contained in that of $F$. In fact, $E = P_{\text{ad}}(\partial)F$ implies

$$\text{sing supp } E := \bigcup_{1 \leq i,j \leq 3} \text{sing supp } E_{ij} \subset \text{sing supp } F,$$

and the analogous inclusion relation holds for the analytic singular supports. Interestingly, as follows from Prop. 4 below, the fundamental matrix is real-analytic outside $K \cap \Xi^*$ and hence, for strictly hyperbolic $R$, no “internal conical refraction” appears in hexagonal media—in contrast with cubic media, cf. [5, p. 53], [32, 6.8, p. 273], [35, 11, p. 138]. Note that this confirms a conjecture going back to Payton [41, Ch. 2, 19, p. 67]: “For this reason it is conjectured that no wave-front lids are necessary” [in hexagonal media]. Later Musgrave [36, p. 579] described the problem in the following way: “An extension of the integral formulation would also help to resolve problems about the existence and nature of the frustum of cone lids on the elastic wave surfaces of some hexagonal media which our findings appear to presage”.

**Proposition 4.** Assume the conditions in (7) and let $P,S,E$ be defined as in Prop. 3 and $K,\Xi^*$ by (9), (10). Let $Q = \det P = W_1 \cdot R$ as in (6) with $W_1(\tau,\xi) = \tau^2$ –
$a_4 \rho^2 - a_5 \xi_3^2, \rho^2 = \xi_1^2 + \xi_2^2,$ and define, furthermore, $\tilde{\xi} = (\xi_2, -\xi_1, 0)^T, W_3(\tau, \xi) = \tau^2 - a_1 \rho^2 - a_5 \xi_3^2,$ and $W_4(\tau, \xi) = \tau^2 - a_5 \rho^2 - a_2 \xi_3^2.$ Then the following holds:

a) $T = P^{\text{ad}} S$ is a matrix of Radon measures on $\mathbb{R}^4 \setminus \{0\}.$

b) $T = \frac{\tilde{\xi} \tilde{\xi}^T}{\rho^2} \text{sign}(\tau) \delta(W_1) + \begin{pmatrix} \xi^2_1 W_4 & \xi \xi_2 W_4 & a_3 \xi_1 \xi_3 W_4 \\ \xi \xi_2 W_4 & \xi^2_2 W_4 & a_3 \xi_2 \xi_3 W_4 \\ a_3 \xi_1 \xi_3 & a_3 \xi_2 \xi_3 & W_3 \end{pmatrix} \text{sign}(\partial_\tau R) \delta(R)$ in $\mathbb{R}^4 \setminus \{0\}.$

(If $R$ is the square of a second-order polynomial, i.e. if $a_3 = 0$ and $a_1 = a_2 = a_5,$ then of course $\delta(R)$ is meaningless and $T = I_3 \text{sign}(\tau) \delta(W_1).$)

c) If $a_2 \neq a_5$ and $Q$ does not contain a square factor, then

$$\text{sing supp } E = \text{sing supp}_A E = K \cap \Xi^T,$$

i.e., the wave-front surface is the dual of the slowness surface and no conical refraction appears.

**Proof.** In addition to the definition of $W_1, W_3, W_4$ in Prop. 4, let us set $W_2(\tau, \xi) = (a_1 - a_4) W_4 + a_2 \xi_3^2.$ Then $R = W_1 W_4 - \rho^2 W_2$ and

$$P^{\text{ad}} = \begin{pmatrix} R + \xi^2_1 W_2 & \xi_1 \xi_2 W_2 & a_3 \xi_1 \xi_3 W_1 \\ \xi_1 \xi_2 W_2 & R + \xi^2_2 W_2 & a_3 \xi_2 \xi_3 W_1 \\ a_3 \xi_1 \xi_3 W_1 & a_3 \xi_2 \xi_3 W_1 & W_3 \end{pmatrix}.$$  

a) According to property (i) in Prop. 3, $S$ is a Radon measure outside the singular points of $Q = W_1 \cdot R.$ These singularities can occur in three ways:

α) In $\xi_1 = \xi_2 = 0, \xi_3 = \pm \frac{\tau}{\sqrt{a_5}},$ $W_1$ and $R$ have common zeros;

β) $W_1$ and $R$ can have further common zeros on circular cones (see Remark 2 to Prop. 2 and Fig. 2, left part);

γ) $R$ has singular points if $a_1 = a_5$ or $a_2 = a_5$ or $[a_3 = 0 \text{ and } (a_1 - a_5)(a_2 - a_5) < 0]$ (see Remark 1 to Prop. 2).

In the neighborhood of such points, $S$ is a distribution of order 1 if just two of the sheets of $Q = 0$ meet, and else is of order two (if e.g. $\xi_1 = \xi_2 = 0, \xi_3 = \pm \frac{\tau}{\sqrt{a_5}}$ and $a_2 = a_5$), cf. the formulas for $S$ in (iii) of Prop. 3. In all these cases, $P^{\text{ad}}$ vanishes at the points in question, and it vanishes of second order if all three sheets intersect. Indeed, for $\rho = 0,$ we have $R = W_1 W_4,$ and hence $P^{\text{ad}} = W_1 \begin{pmatrix} W_4 & 0 & 0 \\ 0 & W_4 & 0 \\ 0 & 0 & W_3 \end{pmatrix}$ vanishes if, furthermore, $\xi_3 = \pm \frac{\tau}{\sqrt{a_5}},$ i.e. if $W_1 = 0.$ Similarly, in the case β), $R = W_1 = 0$ and $\rho \neq 0$ imply $W_2 = 0$ and hence again $P^{\text{ad}} = 0.$ The third case can be checked similarly. Therefore, we infer that $P^{\text{ad}} \cdot S$ is a Radon measure on $\mathbb{R}^4 \setminus \{0\}.$

b) Let us repeat first that, outside the singular points of $Q,$ $S$ is given by

$$S = \frac{\delta(R)}{W_1} \text{sign}(\partial_\tau R) + \frac{\delta(W_1)}{R} \text{sign}(\tau),$$
cf. Prop. 3, (ii). On the other hand, for \( W_1 = 0 \) we have \( R = -\rho^2 W_2 \) and hence \( P^{ad} = -\tilde{\xi} \xi^T W_2 \). This yields

\[
P^{ad} \frac{\delta(W_1)}{R} = \frac{\tilde{\xi} \tilde{\xi}^T}{\rho^2} \delta(W_1) \text{ for } R \neq 0.
\]

Similarly, for \( R = 0 \) and \( W_1 \neq 0 \), we obtain

\[
\frac{W_2}{W_1} = \frac{W_2 W_4}{W_1 W_4} = \frac{W_2 W_4}{W_4} = \frac{W_4}{\rho^2},
\]

and this furnishes

\[
(11) \quad P^{ad} \frac{\delta(R)}{W_1} = \begin{pmatrix}
\frac{\xi_1^2}{\rho} W_4 & \frac{\xi_1 \xi_2}{\rho^2} W_4 & a_3 \xi_1 \xi_3 \\
\frac{\xi_1 \xi_2}{\rho^2} W_4 & \frac{\xi_2^2}{\rho^2} W_4 & a_3 \xi_2 \xi_3 \\
a_3 \xi_1 \xi_3 & a_3 \xi_2 \xi_3 & W_3
\end{pmatrix} \delta(R) \text{ for } W_1 \neq 0.
\]

The formula in b) of the proposition is a consequence of these two identities if we take into account that \( T \) is absolutely continuous with respect to the surface measure supported by \( Q = 0 \). E.g., for singularities of \( Q \) with transversal crossings (in the cases \( \beta \) and \( \gamma \)), this is implied by the following equation in \( D'(\mathbb{R}^2 \setminus \{0\}) \), which is easily verified on test functions:

\[
(\tau - \xi_1) \cdot \delta(\tau^2 - \xi_1^2) \text{sign}(\tau) = \delta(\tau + \xi_1).
\]

c) The formula

\[
E = -F^{-1}(\lim_{\epsilon \to 0} P(\tau - i\epsilon, \xi)^{-1})
\]

and the precise description of the wave-front set of a homogeneous distribution in [26, Thm. 8.1.8, p. 258] imply that \( K \cap \Xi^* \subset \text{sing supp} E \) (compare the reasoning in [47, 2.1, p. 277] and [49, 2., p. 409]).

On the other hand, the explicit formula for \( P^{ad} S \) in b) in conjunction with the representation of \( E \) in formula (8) shows that \( E \) is analytic as long as the plane

\[
\Pi_{(t,x)} := \{ \xi \in \mathbb{R}^3; t + x \xi = 0 \}
\]

neither is tangent to one of the surfaces \( W_1(1, \xi) = 0 \) and \( R(1, \xi) = 0 \) nor does it contain points on the slowness surface with \( \rho = 0 \). If \( R \) is strictly hyperbolic, then \( R(1, \xi) = 0 \) is non-singular and hence the tangency condition is equivalent to \((t, x) \in \Xi^*\) due to the assumption that \( Q \) has no square factor. If \( a_3 = 0 \) or \( a_1 = a_5 \), an additional argument is needed here. This is provided by a further reduction of \( P^{ad} \frac{\delta(R)}{W_1} \) \( \text{sign}(\partial, R) \) in formula (11). For \( a_3 = 0 \), this simply follows from \( R = W_3 W_4 \).

(Alternatively, we could also employ the explicit algebraic expression for \( E \) resulting from the calculation in 5.1 below.) If \( a_1 = a_5 \), then

\[
R = W_5^2 - a_3^2 \xi_3^2 \rho^2 - \left( \frac{a_1 - a_2}{2} \right)^2 \xi_3^4 \quad \text{with} \quad W_5 := \tau^2 - a_1 \rho^2 - \frac{a_1 + a_2}{2} \xi_3^2
\]
and the surface \( R(1, \xi) = 0 \) is singular along the circle \( \xi_3 = 0, \ \rho = 1/\sqrt{a_1} \). But upon setting \( f(\xi) := a_3^2 \rho^2 + \left( \frac{a_1 - a_2}{2} \right)^2 \xi_3^2 \), we can decompose \( R \) near this circle, i.e. \( R = (W_5 + \xi_3 \sqrt{f})(W_5 - \xi_3 \sqrt{f}) \) and conclude that

\[
\xi_3 \delta(R) \operatorname{sign}(\partial_{\tau} R) = \frac{\operatorname{sign}(\tau)}{2\sqrt{f}} \cdot [\delta(W_5 - \xi_3 \sqrt{f}) - \delta(W_5 + \xi_3 \sqrt{f})].
\]

In a similar way, the products \( W_3 \delta(R) \) and \( W_4 \delta(R) \) which appear in the right-hand side of (11) can be expressed by \( \delta(W_5 \pm \xi_3 \sqrt{f}) \), and therefore \( E \) remains regular if \( \Pi_{(t,x)} \) intersects \( R(1,\xi) = 0 \) without being tangential to \( W_5(1,\xi) \pm \xi_3 \sqrt{f}(\xi) = 0 \).

Finally, the conditions \( \xi \in \Pi_{(t,x)}, \ (1,\xi) \in \Xi, \ \rho = 0 \) imply \( t = \pm \frac{x_3}{\sqrt{a_5}} \) or \( t = \pm \frac{x_3}{\sqrt{a_2}} \), and at such a point \( F \) and a fortiori \( E \) are analytic unless it belongs to \( K \cap \Xi^\tau \) according to the description of \( \text{sing supp } F \) preceding Prop. 4. This shows

\[
\text{sing supp}_A E \subset K \cap \Xi^\tau
\]

and thus completes the proof.

**Remarks.** 1) In the case of \( a_2 = a_5 \), i.e. \( c_{33} = c_{44} \), which has been excluded in Prop. 4 c), the operator \( R(\partial) \) is not strictly hyperbolic and the conical points on the slowness surface of \( R(\partial) \) produce “lids” of conical refraction. This fact had been conjectured in [36] (see the quotation prior to Prop. 4) and was first established by R. G. Payton in the special case of \( a_1 = a_2 = a_5 \), cf. [42].

Let us investigate now the case \( a_2 = a_5 \) more in detail. For \( a_2 = a_5 \), the points \( (0,0, \pm \frac{1}{\sqrt{a_2}}) \) are triple points of the slowness surface \( \{ \xi \in \mathbb{R}^3; Q(1,\xi) = 0 \} \) (with \( Q = \det P \) as before), see Fig. 3, left part, and the localizations \( Q_\eta \) with respect to \( \eta = \tau(1,0,0, \pm \frac{1}{\sqrt{a_2}}) \) are given by

\[
Q_\eta(\partial) = 2\tau^2 (\partial_t \mp \sqrt{a_2} \partial_3) \cdot [4(\partial_t \mp \sqrt{a_2} \partial_3)^2 - \frac{a_3^2}{a_2} \Delta_2].
\]

By [1], the set

\[
M := \bigcup_{\pm} \Gamma(Q_{(1,0,0, \pm 1/\sqrt{a_2})}^* = \left\{ (t,x) \in \mathbb{R}^4; |x_3| = t\sqrt{a_2}, |x'| \leq \frac{|a_3| t}{2 \sqrt{a_2}} \right\}
\]

is contained in the singular support of the fundamental solution of \( Q(\partial) \). The two flat lids represented by \( M \) yield the two horizontal lines in Fig. 3 on the right part, where we have depicted the slowness and the wave-front surface for a fictitious material with the same elastic constants as in \( \text{TiB}_2 \) except for \( a_5 = a_2 = 440 \).

In order to prove that conical refraction takes place in the case of \( a_2 = a_5 \), we still have to verify that \( M \) is contained in \( \text{sing supp}_A E \), which set can be a proper subset of \( \text{sing supp}_A F \) (compare the discussion before Prop. 4). But this follows from Prop. 6 below: If we had \( M \not\subset \text{sing supp}_A E \), then \( E_{33} \) would vanish in a cuspidal region \( B \).
around the \( x_3 \)-axis by analytic continuation from \( \mathbb{R}^4 \setminus K \) (and since \((E_1)_{33} = 0\)), in contradiction with the explicit formula for the values of \( E_{33} \) on the \( x_3 \)-axis in Prop. 6.

In contrast to the case of \( a_2 = a_5 \), for \( a_1 = a_5 \), the \textit{cylindrical lid} which appears in the singular support of the fundamental solution \( F \) of \( Q = \det P \), is—as was shown in Prop. 4—not present in \( \text{sing supp} E \). In the special case of \( a_1 = a_2 = a_5 \) this was proven in [42] for \( E_{33} \) and conjectured for \( E_{31}, E_{32} \). In Fig. 4, we depict the slowness and wave-front surface for \( a_1 = a_5 \). The vertical broken lines (which represent the cylindrical lid) are contained in \( \text{sing supp} F \) but not in \( \text{sing supp} E \). Note also that the support of \( E \) is bounded by a non-convex cone.

---

Figure 3: Slowness and wave-front surface for \( c_{33} = c_{44} \)
Conical refraction along two flat circular lids

Figure 4: Slowness and wave-front surface for \( c_{11} = c_{44} \)
No conical refraction along the cylindrical lid
2) The open set $K \setminus \Xi$ consists of four different types of points, and we denote the respective sets by $A, B, C, D$. The convex inner region $A$ of $K$ is the component of $K \setminus \Xi$ containing $(1,0,0,0)$. The fundamental matrix $E$ vanishes in $A$ by formula (8) in Prop. 3, since
\[
\text{supp} (S(1, \xi)) \cap \Pi_{(t,x)} = \emptyset
\]
if $(t, x) \in A$. (The set $A$ is a Petrovsky lacuna in the sense of [1, p. 185], [27, Thm. 12.6.6, p. 132].) If the outer cone of $R = 0$ is non-convex (as is the case in Figs. 1, 2, but not in general), then there appear cuspidal regions, cf. Figs. 1, 2, right side. They consist of those points $(t, x)$ for which $\Pi_{(t,x)}$ intersects the non-convex part of $R(1, \xi) = 0$ along two curves. If $(t, x)$ belongs to the set $C$, then this intersection consists of just one connected component. For $(t, x) \in D$, the plane $\Pi_{(t,x)}$ intersects the ellipsoid $W_1(1, \xi) = 0$, but not the quartic $R(1, \xi) = 0$.

4. The fundamental matrix of hexagonal elastodynamics: Quantitative aspects

In the next proposition, we shall reduce the three-dimensional integral in formula (8) to a one-fold integral with respect to $\xi_3$. This remaining integral is a complete Abelian integral pertaining to a Riemannian surface of genus 3 (for general $t, x, a_1, \ldots, a_5$).

**Proposition 5.** Let $P(\partial) = \partial_t^2 I_3 + A(\nabla)$ be the system of hexagonal elastodynamics as given by formula (3) and suppose that $a_1, a_2, a_4, a_5$ are positive and that $|a_3| < a_5 + \sqrt{a_1a_2}$. Then the unique fundamental matrix $E$ of $P(\partial)$ with support in $H_+$ is given by $E = E_1 + E_2 + E_3 + E_4$ where
\[
E_1 = \frac{1}{4\pi a_4 \sqrt{a_5} |x'|^2} \left( \begin{array}{ccc}
\frac{x_2^2}{2} & -x_1 x_2 & 0 \\
-x_1 x_2 & x_1^2 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \delta \left( t - \frac{|x'|^2}{a_4} + \frac{x_3^2}{a_5} \right)
\]
\[
+ \frac{1}{4\pi \sqrt{a_5} |x'|^4} \left( \begin{array}{ccc}
x_2^2 & -x_1 x_2 & 0 \\
x_1 x_2 & x_1^2 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \left[ Y \left( t - \frac{|x_3|^2}{\sqrt{a_5}} \right) \right] - \left[ Y \left( t - \sqrt{\frac{|x'|^2}{a_4} + \frac{x_3^2}{a_5}} \right) \right],
\]
\[
E_2 = \frac{(a_1 a_5)^{3/2}}{8\pi^2} Y(t) \left( \begin{array}{ccc}
\frac{x_1}{|x'|^2} & \frac{x_2}{|x'|^2} & 0 \\
\frac{x_2}{|x'|^2} & \frac{x_3}{|x'|^2} & 0 \\
0 & 0 & 0 \\
\end{array} \right) \sum_{j=1}^{2} \epsilon_j \int_{-s_0}^{s_0} (t + x_3 s) \times
\]
\[
\times \left( \frac{1 - a_2 s^2}{a_1^2 a_5} - a_6 s^2 + (-1)^{j-1} \sqrt{D(s)} \right) \frac{Y(\mu_j(s))}{\sqrt{\mu_j(s)}} \frac{ds}{\sqrt{D(s)}},
\]
\[
E_3 = -\frac{a_3 \sqrt{a_1 a_5}}{8\pi^2} Y(t) \left( \begin{array}{ccc}
0 & 0 & \frac{1}{\partial_2} \\
0 & 0 & \partial_2 \\
\partial_1 & \partial_2 & 0 \\
\end{array} \right) \sum_{j=1}^{2} \epsilon_j \int_{-s_0}^{s_0} s \frac{Y(\mu_j(s))}{\sqrt{\mu_j(s)}} \frac{ds}{\sqrt{D(s)}},
\]
with the abbreviations

\[
a_6 = \frac{1}{2}(a_1a_2 + a_5^2 - a_3^2), \quad a_7 = (a_5^2 - a_1a_2)(a_1 - a_5) + a_3^2(a_1 + a_5), \quad |x'| = \sqrt{x_1^2 + x_2^2},
\]

\[
s_0 = \begin{cases} 
\frac{1}{\sqrt{a_2}} : a_3^2 \leq a_5(a_5 - a_2), \\
\frac{1}{\sqrt{a_5}} : a_3^2 \leq a_1(a_2 - a_5), \\
\left[ a_7 + 2|a_3|\sqrt{a_1a_5}\sqrt{a_3^2 - (a_1 - a_5)(a_2 - a_5)} \right]^{1/2} : \text{else},
\end{cases}
\]

\[
\epsilon_j = \begin{cases} 
-1 : (t, x) \in B \text{ (see Remark 2 to Prop. 4)}, \\
(-1)^j : \text{else},
\end{cases}
\]

\[
D(s) = \frac{1}{4}(a_1 + a_5 - 2a_6s)^2 - a_1a_5(1 - a_2s^2)(1 - a_5s^2),
\]

\[
\mu_{1,2}(s) = -a_1a_5(t + 3s)^2 + \frac{a_1 + a_5}{2} |x'|^2 - a_6|x'|^2s^2 \pm |x'|^2\sqrt{D(s)}, \quad \mu_1 \geq \mu_2.
\]

**Proof.** a) When we insert the formula for \( T = P^{ad}S \) in Prop. 4 b) into the representation of \( E \) in (8), we obtain two parts related to \( \delta(W_1) \) and to \( \delta(R) \), respectively. Let us first calculate explicitly the simple part containing \( \delta(W_1) \), i.e.

\[
E_1 := -\frac{Y(t)}{4\pi^2} \partial_t \int_{\mathbb{R}^3} \frac{\xi\xi^T}{\rho^2} \delta(W_1(1, \xi)) \delta(t + x\xi) \, d\xi.
\]

Putting \( \hat{\nabla} = \begin{pmatrix} \partial_2 \\ -\partial_1 \\ 0 \end{pmatrix} \) we infer

\[
E_1 = -\frac{Y(t)}{4\pi^2} \hat{\nabla} \int_{\mathbb{R}^3} \frac{\xi\xi^T}{\rho^2} \delta(W_1(1, \xi)) \delta(t + x\xi) \, d\xi.
\]

In order to transform the last integral into an integral over the ellipsoid \( W_1(1, \xi) = 0 \) oriented by the Kronecker–Leray form

\[
\gamma(\xi) = \xi_1 d\xi_2 \wedge d\xi_3 - \xi_2 d\xi_1 \wedge d\xi_3 + \xi_3 d\xi_1 \wedge d\xi_2,
\]

we use \( dW_1(1, \xi) \wedge \gamma = -2 d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \) on \( W_1(1, \xi) = 0 \), which yields

\[
\int_{\mathbb{R}^3} \phi(\xi) \delta(W_1(1, \xi)) \, d\xi = \frac{1}{2} \int_{W_1(1, \xi) = 0} \phi(\xi) \gamma(\xi)
\]
for a test function $\phi$. Therefore, Prop. 5 in [38, p. 344] furnishes

$$E_1 = -\frac{Y(t)a_4}{8\pi^2} \nabla \int_{W_1(1,\xi)=0} \frac{\xi^T}{1 - a_5\xi_3^2} \delta(t + x\xi) \gamma(\xi)$$

$$= \frac{Y(t)}{8\pi \sqrt{a_5}} \frac{2^T}{|x'|^2} Y \left( \frac{|x'|^2}{a_4} + \frac{x_2^2}{a_5} - t^2 \right) \cdot \left[ \text{sign}(t + \frac{x_3}{\sqrt{a_5}}) + \text{sign}(t - \frac{x_3}{\sqrt{a_5}}) \right]$$

$$= \frac{Y(t) - \frac{|x_3|}{\sqrt{a_5}}}{4\pi \sqrt{a_5} |x'|^4} \left( \begin{array}{ccc} x_1^2 - x_2^2 & 2x_1x_2 & 0 \\ 0 & 2x_1x_2 & x_1^2 - x_2^2 \\ 0 & 0 & 0 \end{array} \right) +$$

$$+ \frac{\delta(t - \sqrt{\frac{|x'|^2}{a_4} + \frac{x_2^2}{a_5}})}{4\pi a_4 \sqrt{a_5} t |x'|^2} \left( \begin{array}{ccc} x_2^2 - x_1x_2 & -x_1x_2 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

b) It remains to determine the terms containing $\delta(R)$. According to the structure of the matrix factor of $\delta(R)$ in Prop. 4 b), we set

$$E - E_1 = E_2 + E_3 + E_4,$$

where we abbreviate $\sigma(\xi) := \text{sign}(\partial_\nu R(1,\xi))$ and define

$$E_2 := -\frac{Y(t)}{8\pi^2} \int_{\mathbb{R}^3} \frac{W_4(1,\xi)}{\rho^2} \left( \begin{array}{ccc} \xi_1^2 & \xi_1\xi_2 & 0 \\ \xi_1\xi_2 & \xi_2^2 & 0 \\ 0 & 0 & 0 \end{array} \right) \delta(R(1,\xi)) \sigma(\xi) \text{sign}''(t + x\xi) d\xi$$

$$= -\frac{Y(t)}{8\pi^2} \left( \begin{array}{ccc} \partial_1^2 & \partial_1\partial_2 & 0 \\ \partial_1\partial_2 & \partial_2^2 & 0 \\ 0 & 0 & 0 \end{array} \right) \int_{\mathbb{R}^3} \frac{1 - a_5\rho^2 - a_2\xi_3^3}{\rho^2} \delta(R(1,\xi)) \sigma(\xi) \text{sign}(t + x\xi) d\xi,$$

and

$$E_3 := -\frac{a_3 Y(t)}{4\pi^2} \left( \begin{array}{ccc} 0 & 0 & \partial_1 \\ 0 & 0 & \partial_2 \\ \partial_1 & \partial_2 & 0 \end{array} \right) \int_{\mathbb{R}^3} \xi_3 \delta(R(1,\xi)) \sigma(\xi) \delta(t + x\xi) d\xi,$$

$$E_4 := -\frac{Y(t)}{4\pi^2} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_1 \end{array} \right) \int_{\mathbb{R}^3} (1 - a_1\rho^2 - a_5\xi_3^3) \delta(R(1,\xi)) \sigma(\xi) \delta(t + x\xi) d\xi.$$

Because the integrands in the above three integrals are rotationally symmetric with respect to $\xi_1, \xi_2$, we can set therein $x_2 = 0$ and $x_1 = |x'|$ and first integrate with respect to $\xi_2$. We use the elementary formulas

$$\int_{\mathbb{R}} \phi(s) \delta(as^2 - b) ds = \frac{Y(ab)}{\sqrt{ab}} \phi \left( \sqrt{\frac{b}{a}} \right)$$
and

\begin{equation}
\int_{\mathbb{R}} \phi(s) \delta(as^4 + 2bs^2 + c) \, ds = \frac{Y(b^2 - ac)}{2 \sqrt{b^2 - ac}} \sum_{j=1}^{2} \frac{Y(\lambda_j)}{\sqrt{\lambda_j}} \phi(\sqrt{\lambda_j}),
\end{equation}

which are valid for an even continuous function \( \phi \) and \( a, b, c \in \mathbb{R} \setminus \{0\} \), \( ac \neq b^2 \), and

\[ \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - ac}}{a}. \]

Therefore,

\begin{equation}
\int_{\mathbb{R}} \phi(\xi_2) \delta(R(1, \xi)) \, d\xi_2 = \frac{Y(D)}{2 \sqrt{D}} \sum_{j=1}^{2} \frac{Y(\lambda_j)}{\sqrt{\lambda_j}} \phi(\sqrt{\lambda_j}),
\end{equation}

wherein

\[ R(1, \xi) = a_1 a_5 \rho^4 - (a_1 + a_5 - 2a_6 \xi_3^2) \rho^2 + (1 - a_2 \xi_3^2)(1 - a_5 \xi_3^2) \]

and hence

\[ D(\xi_3) = \frac{1}{4}(a_1 + a_5 - 2a_6 \xi_3^2)^2 - a_1 a_5(1 - a_2 \xi_3^2)(1 - a_5 \xi_3^2) \]

and

\[ \lambda_{1,2} = \frac{1}{a_1 a_5} (-a_1 a_5 \xi_1^2 + \frac{a_1 + a_5}{2} - a_6 \xi_3^2 \pm \sqrt{D(\xi_3)}). \]

Furthermore,

\[ \phi(\xi_2) = \frac{1}{\xi_1^2 + \xi_2^2} \frac{1 - a_5(\xi_1^2 + \xi_2^2) - a_2 \xi_3^3}{\xi_1^2 + \xi_2^2} \sigma(\xi) \text{ or } \phi(\xi_2) = \sigma(\xi) \]

or \( \phi(\xi_2) = (1 - a_1(\xi_1^2 + \xi_2^2) - a_5 \xi_3^3) \sigma(\xi) \)

in the respective three integrals representing \( E_2, E_3, E_4 \).

Evidently, \( \int_{\mathbb{R}} \phi(\xi_2) \delta(R(1, \xi)) \, d\xi_2 \) vanishes if

\[ |\xi_3| > s_0 := \max\{s \in \mathbb{R}; \exists(\xi_1, \xi_2) \in \mathbb{R}^2 : R(1, \xi_1, \xi_2, s) = 0\} \]

and a straightforward calculation leads to the formula for \( s_0 \) in Prop. 5. Thus \( s_0 \) is the maximal abscissa in Figs. 1, 2, left part, and \( D(\xi_3) \geq 0 \) for \( |\xi_3| \leq s_0 \). Note also that \( \lambda_1 \) and \( \lambda_2 \) coincide for \( \xi_3 = s_0 > \max(\frac{1}{\sqrt{a_2}}, \frac{1}{\sqrt{a_5}}) \) and hence \( D(s_0) = 0 \) in this case.

Next we integrate with respect to \( \xi_1 \) by using the factor

\[ \delta(t + x\xi) = \delta(t + |x'|\xi_1 + x_3\xi_3) \],
which amounts to replacing \( \xi_1 \) by \( -\frac{t+x_3 \xi_3}{|x'|} \) and to dividing by \( |x'| \). (In the integral for \( E_2 \), this \( \delta \)-factor is produced by the formula

\[
\begin{pmatrix}
\partial_1^2 & \partial_1 \partial_2 & 0 \\
\partial_1 \partial_2 & \partial_2^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\text{sign}(t+|x'|\xi_1+x_3\xi_3) = 2 \begin{pmatrix}
\partial_1 \frac{\tau}{|x'|} & \partial_1 \frac{x_2}{|x'|} & 0 \\
\partial_2 \frac{\tau}{|x'|} & \partial_2 \frac{x_2}{|x'|} & 0 \\
0 & 0 & 0
\end{pmatrix} \xi_1 \delta(t+|x'|\xi_1+x_3\xi_3).
\]

Let us next determine the sign factor \( \sigma(\xi) \). With the notations of Prop. 3, we obtain (for points where \( R \) is non-singular)

\[
R(\tau, \xi) = (\tau^2 - f_2(\xi))(\tau^2 - f_3(\xi)) \quad \text{with } 0 < f_2(\xi) < f_3(\xi)
\]

and thus

\[
\sigma(\xi) = \text{sign}(\partial_\tau R)(1, \xi) = \text{sign}(2 - f_2(\xi) - f_3(\xi)) = \begin{cases} -1 : f_2(\xi) = 1 \\ 1 : f_3(\xi) = 1, \end{cases}
\]

i.e. \( \sigma(\xi) = \{ -1 \} \) on the \{outer\} sheet of the part \( R(1, \xi) = 0 \) of the slowness surface.

If \( (t, x) \in B \), then the plane \( \Pi_{(t, x)} = \{ \xi \in \mathbb{R}^2; t + x_\xi = 0 \} \) intersects the outer sheet of \( R(1, \xi) = 0 \) along two curves and there \( \sigma(\xi) = -1 \). On the other hand, if \( (t, x) \notin B \), then \( \Pi_{(t, x)} \) intersects the outer sheet of \( R(1, \xi) = 0 \), where \( \sigma(\xi) = -1 \), at most along one curve, and this curve is obtained by taking the larger root \( \lambda_1 \) of \( R(1, \xi) = 0 \) with respect to \( \xi_2^2 \). If \( \Pi_{(t, x)} \) also intersects the inner sheet of \( R(1, \xi) = 0 \), then \( \sigma(\xi) = 1 \) everywhere on the respective curve. Thus we conclude that we have \( \sigma(\xi) = (-1)^j \) for \( (t, x) \notin B \) and \( \sigma(\xi) = \epsilon_j \) in general, \( \epsilon_j \) being defined as in Prop. 5.

Hence denoting the remaining integration variable \( \xi_3 \) by \( s \) and using

\[
\lambda_j \left( -\frac{t+x_3 s}{|x'|}, s \right) = \frac{\mu_j(s)}{a_1 a_5 |x'|^2}
\]

we obtain finally the following for \( E_3 \):

\[
E_3 = -\frac{a_3 \sqrt{a_1 a_5}}{8\pi^2} Y(t) \begin{pmatrix}
0 & 0 & \partial_1 \\
0 & 0 & \partial_2 \\
\partial_1 & \partial_2 & 0
\end{pmatrix} \sum_{j=1}^2 \epsilon_j \int_{s_0}^{s_0} s \frac{Y(\mu_j(s))}{\sqrt{\mu_j(s)}} \cdot \frac{ds}{\sqrt{D(s)}}.
\]

The integral in the formula for \( E_4 \) yields

\[
\int_{\mathbb{R}^3} (1 - a_1 \rho^2 - a_5 \xi_3^2) \delta(R(1, \xi)) \sigma(\xi) \delta(t + x \xi) \, d\xi =
\]

\[
= \frac{1}{2} \sum_{j=1}^2 \epsilon_j \int_{\mathbb{R}^2} (1 - a_1 \xi_1^2 + \lambda_j) - a_5 \xi_3^2 \frac{Y(D(\xi_3))}{\sqrt{D(\xi_3)}} \cdot \frac{Y(\lambda_j)}{\lambda_j} \, d\xi_1 d\xi_3.
\]

Because of

\[
1 - a_1 (\xi_1^2 + \lambda_j) - a_5 \xi_3^2 = \frac{1}{a_5} \left( \frac{a_5 - a_1}{2} + (a_6 - a_5^2) \xi_3^2 + (-1)^j \sqrt{D(\xi_3)} \right),
\]
we arrive at
\[
E_4 = -\frac{\sqrt{a_1}}{8\pi^2\sqrt{a_5}} Y(t) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_t \end{array} \right) \frac{2}{\partial_t} \sum_{j=1}^{s_0} \left( \frac{a_5 - a_1}{2} + (a_6 - a_2^3)s^2 + (-1)^j \sqrt{D(s)} \right) \times \frac{Y(\mu_j(s))}{\sqrt{\mu_j(s)}} \cdot \frac{ds}{\sqrt{D(s)}}.
\]

The remaining term \( E_2 \) is treated similarly. \( \square \)

**Remarks.** 1) The integrals in \( E_2, E_3, E_4 \) are Abelian integrals of the genus \( g = 3 \) for generic \( a_j, t, x \). In fact, the Riemannian surface \( S \) defined by \( w^2 = \mu_j(s) \), i.e. \( S = \{(s, w); R(1, (t + x_3 s)/|x'|, w/(\sqrt{a_1 a_5}|x'|), s) = 0\} \), can be viewed as a covering by \( n = 4 \) sheets over the \( s \)-plane ramified over the four zeros of \( D(s) \) (where \( \mu_1 \) and \( \mu_2 \) coincide) and over the four zeros of \( \mu_j \). The degree of the branch locus is \( b = 8 + 4 = 12 \) and the Riemann–Hurwitz formula implies \( g = -n + 1 + \frac{b}{2} = 3 \), cf. [21, p. 216].

For generic \( t, x \), the genus \( g \) is lowered in the following two cases only:

a) \( D(s) \) is a complete square. This happens iff the discriminant \( a_1 a_5 a_2^2 (a_3^2 - (a_1 - a_5)(a_2 - a_5)) \) of the quadratic polynomial \( D(\sqrt{s}) \) vanishes, i.e., iff \( a_3 = 0 \) or \( a_2^3 = (a_1 - a_5)(a_2 - a_5). \) In these cases \( g = 0 \) and the integrals for \( E \) yield algebraic functions, see 5.1, 5.2 below.

b) \( D(s) \) has the factor \( s^2 \). This happens iff \( D(0) = 0 \), i.e. iff \( a_1 = a_5 \). Then the algebraic curve \( S \) has branch points over the two simple zeros of \( D \) and over the four zeros of \( \mu_j \). The genus of \( S \) is therefore \( g = 1 \) and hence \( E \) can be expressed by elliptic integrals, see 5.3 below.

2) Let the sets \( K \) and \( A, B, C, D \) be defined as in (9) and in Remark 2 to Prop. 4, respectively. If \((t, x) \in A\), then \( \text{supp}(S(1, \xi)) \cap \Pi_{t, x} = \emptyset \) and therefore \( \mu_j(s) \) is negative for \(|s| \leq s_0 \). This means that each \( E_j \) in Prop. 5 vanishes trivially. On the other hand, if \( t \geq 0 \) and \((t, x) \in \mathbb{R}^3 \setminus K\), then \( \Pi_{t, x} \) intersects the set
\[
\text{supp}(S(1, \xi)) = \{\xi \in \mathbb{R}^3; \det P(1, \xi) = 0\}
\]
in three curves and \( E_1 + E_2, E_3, E_4 \) vanish due to the summation with respect to \( j \). If \((t, x) \in D\), then \( E_2 = E_3 = E_4 = 0 \) and \( E = E_1 \) is a sum of a delta function and of a rational function.

5. The fundamental matrix in the cases of genus 0 or 1

As has been stated in the Remark 1 to Prop. 5, the integrals representing \( E \) are elementary integrals for general \( t, x \) (i.e. the genus of the corresponding algebraic curve is 0) in the two cases \( a_3 = 0 \) and \( a_2^3 = (a_1 - a_5)(a_2 - a_5) \). These conditions are also equivalent to the decomposability of the fourth-order factor \( R \) of \( \det P \) into two “aligned” quadrics (i.e. with common eigenvectors), cf. [41, p. 96], [10, (1.1), (1.2),
p. 589], [6], [38, p. 320]. Though these two conditions are not plausible from the physical viewpoint (cf. [9, p. 44]), the corresponding special cases are valuable for reasons of control, since therein the fundamental matrix can be represented by means of elementary functions. In Sections 5.1, 5.2, we shall derive explicit formulas for $E$ from our general formula in Prop. 5, and we shall obtain complete agreement with the results in [38, Props. 1, 2, pp. 336, 339], which had been calculated directly there and earlier, by different methods, in [41, p. 96], [6].

In Section 5.3, we will perform explicitly the reduction of the integrals in Prop. 5 to elliptic integrals under the constraint $a_1 = a_5$, which leads to the reduced genus 1 (see Remark 1 to Prop. 5). In this case we obtain

$$R(\tau, \xi) = (\tau^2 - a_1 \rho^2 - \sqrt{a_1 a_2} \xi_3^2)^2 - (\sqrt{a_1} - \sqrt{a_2})^2 \xi_3^2 - (a_3^2 - a_1(\sqrt{a_1} - \sqrt{a_2})) \rho^2 \xi_3^2.$$  

If, additionally, $a_1 = a_2$ holds (the case considered in [42]; see also Remark 1 to Prop. 4), then $R(\partial)$ factors into two pseudodifferential operators whose slowness surfaces are given by ellipses in the $\xi_1 \xi_3$-plane; if $a_3^2 = a_1(\sqrt{a_1} - \sqrt{a_2})^2$, then $R(\partial)$ factors into two “non-aligned” wave operators.

**5.1. The case of $a_3 = 0$, i.e. $c_{13} + c_{44} = 0$**

Let us investigate first the case of $a_3 = 0$, where $\det P = W_1 W_3 W_4$ with $W_j$ defined as in Prop. 4 b). Since both sheets of $R = 0$, i.e. the ellipsoids $W_3 = 0$ and $W_4 = 0$, are convex, the cuspidal regions denoted by $B$ do not occur, and $\epsilon_j = (-1)^j$. If $D$ and $\mu_j$ are defined as in Prop. 5, then we obtain from $a_3 = 0$ that

$$D(s) = \frac{1}{4} [(a_5^2 - a_1 a_2)s^2 + a_1 - a_5]^2$$

and $\mu_1 = p_\epsilon$, $\mu_2 = p_{-\epsilon}$, where

$$\epsilon := \text{sign}((a_5^2 - a_1 a_2)s^2 + a_1 - a_5),$$

$$p_\pm := \left\{ \begin{array}{l}
 a_1 [|x'|^2 (1 - a_2 s^2) - a_5 (t + x_3 s)^2] \\
 a_5 [|x'|^2 (1 - a_5 s^2) - a_1 (t + x_3 s)^2].
\end{array} \right.$$  

Furthermore, for the factors in the integrals of $E_2$ and $E_4$, we have

$$\frac{1 - a_2 s^2}{a_1 + a_5} - \frac{a_1 a_2 - a_5^2}{2} s^2 + (-1)^{j-1} \sqrt{D(s)} - \frac{1}{a_1} = \frac{2\epsilon \sqrt{D(s)} Y(\epsilon(-1)^{j-1})}{a_1 a_5 (1 - a_5 s^2)}$$

and

$$\frac{a_5 - a_1}{2} + \frac{a_1 a_2 - a_5^2}{2} s^2 + (-1)^j \sqrt{D(s)} = 2(-1)^j \sqrt{D(s)} Y(\epsilon(-1)^{j-1})$$
and hence we obtain (with $s_0 = \max\left(\frac{1}{\sqrt{a_5}}, \frac{1}{\sqrt{a_6}}\right)$)

$$E_2 = \frac{(a_1a_5)^{3/2}}{8\pi^2} Y(t) \left( \begin{array}{ccc} \partial_1 \frac{x_1}{|x|^2} & \partial_1 \frac{x_2}{|x|^2} & 0 \\ \partial_2 \frac{x_1}{|x|^2} & \partial_2 \frac{x_2}{|x|^2} & 0 \\ 0 & 0 & 0 \end{array} \right) \int_{-s_0}^{s_0} \frac{t + x_3 s}{a_1a_5(1-a_5s^2)} \times$$

$$\times \left[ -2\epsilon Y(-\epsilon) \frac{Y(p_\epsilon)}{\sqrt{p_\epsilon}} + 2\epsilon Y(\epsilon) \frac{Y(-p_\epsilon)}{\sqrt{-p_\epsilon}} \right] ds$$

$$= \frac{\sqrt{a_1}}{4\pi^2} Y(t) \left( \begin{array}{ccc} \partial_1 \frac{x_1}{|x|^2} & \partial_1 \frac{x_2}{|x|^2} & 0 \\ \partial_2 \frac{x_1}{|x|^2} & \partial_2 \frac{x_2}{|x|^2} & 0 \\ 0 & 0 & 0 \end{array} \right) \int_{-s_0}^{s_0} \frac{t + x_3 s}{1-a_5s^2} \cdot \frac{Y(|x'|^2(1-a_5s^2) - a_1(t+x_3s)^2)}{\sqrt{|x'|^2(1-a_5s^2) - a_1(t+x_3s)^2}} ds.$$

In order to evaluate the last elementary integral, we use the next lemma, cf. [18, p. 116, ex.], [34, 5.4.2, Thm. 2, p. 188], [22, 212.5].

**Lemma.** Let $p(x) = ax^2 + bx + c$, $l(x) = dx + e$, $a, b, c, d, e, \alpha \in \mathbb{R}$ with $a < 0$, $\alpha > 0$, $p(\alpha) < 0$, $p(-\alpha) < 0$ and $\exists x \in \mathbb{R} : p(x) > 0$. Then

$$\int_{-\infty}^{\infty} \frac{l(x)}{\alpha^2 - x^2} \cdot \frac{Y(p(x))}{\sqrt{p(x)}} dx = \frac{\pi}{2\alpha} \left( \frac{l(\alpha)}{\sqrt{-p(\alpha)}} + \frac{l(-\alpha)}{\sqrt{-p(-\alpha)}} \right).$$

If we apply the Lemma to the integral for $E_2$, this yields

$$Y(t) \int_{-s_0}^{s_0} \frac{t + x_3 s}{1-a_5s^2} \cdot \frac{Y(|x'|^2(1-a_5s^2) - a_1(t+x_3s)^2)}{\sqrt{|x'|^2(1-a_5s^2) - a_1(t+x_3s)^2}} ds$$

$$= Y \left( \frac{|x'|^2}{a_1} + \frac{x_3^2}{a_5} - t^2 \right) \cdot \frac{\pi}{\sqrt{a_1a_5}} Y \left( t - \frac{|x_3|}{\sqrt{a_5}} \right).$$

and thus

$$E_2 = \frac{1}{4\pi \sqrt{a_5}} \left( \begin{array}{ccc} \partial_1 \frac{x_1}{|x|^2} & \partial_1 \frac{x_2}{|x|^2} & 0 \\ \partial_2 \frac{x_1}{|x|^2} & \partial_2 \frac{x_2}{|x|^2} & 0 \\ 0 & 0 & 0 \end{array} \right) \left[ Y \left( t - \frac{|x_3|}{\sqrt{a_5}} \right) - Y \left( t - \sqrt{\frac{|x'|^2}{a_1} + \frac{x_3^2}{a_5}} \right) \right].$$

(Note that the differentiation symbols $\partial_1, \partial_2$ act on all the following terms, in particular also on the Heaviside functions.)

Similarly,

$$E_{33} = (E_4)_{33} = -\frac{Y(t) \sqrt{a_1}}{4\pi^2 \sqrt{a_5}} \partial_t \int_{-s_0}^{s_0} \frac{Y(p_\tau)}{\sqrt{p_\tau}} ds = -\frac{Y(t)}{4\pi^2 \sqrt{a_5}} \partial_t \frac{\pi Y \left( \frac{|x'|^2}{a_5} + \frac{x_3^2}{a_2} - t^2 \right)}{\sqrt{a_2 |x'|^2 + a_5 x_3^2}}$$

$$= \frac{1}{4\pi a_5 \sqrt{a_2}} \delta \left( t - \sqrt{\frac{|x'|^2}{a_5} + \frac{x_3^2}{a_2}} \right).$$
Adding \( E_1, E_2, E_3 = 0 \), and \( E_4 \) yields the fundamental matrix \( E \) which is explicitly stated in \([38, \text{Prop. 1, p. 337}]\). In a slightly less explicit form, \( E \) is also contained in \([6, (6.4), \text{p. 669}]\).

### 5.2. The case of \( a_3^2 = (a_1-a_5)(a_2-a_5) \), i.e. \( (c_{13}+c_{44})^2 = (c_{11}-c_{44})(c_{33}-c_{44}) \)

This second case is slightly more complicated. For simplicity, let us assume in addition that \( a_1 > a_2 > a_5 > 0 \) as in \([41, \text{p. 96}]\) and in \([38, 4.2, \text{p. 337}]\). The equation \( a_3^2 = (a_1-a_5)(a_2-a_5) \) leads to the factorization
\[
R = (\tau^2 - a_1(\xi_1^2 + \xi_2^2) - a_2\xi_3^2)(\tau^2 - a_5|\xi|^2).
\]

Again \( \epsilon_j = (-1)^j \) since the cuspidal regions \( B \) do not appear. Furthermore, if \( D \) and \( \mu_j \) are as in Prop. 5, then
\[
D(s) = \frac{1}{4}[a_5(a_1-a_2)s^2 + a_5 - a_1]^2
\]
and \( \mu_1 = p_\epsilon, \mu_2 = p_{-\epsilon} \), where
\[
\epsilon := \text{sign}(a_5(a_1-a_2)s^2 + a_5 - a_1),
\]
\[
p_{\pm} := \begin{cases} a_5(|x'|^2(1-a_2s^2) - a_1(t+x_3s)^2) \\ a_1(|x'|^2(1-a_5s^2) - a_5(t+x_3s)^2). \end{cases}
\]

This implies for the factors in the integrals of \( E_2 \) and \( E_4 \)
\[
\frac{1-a_2s^2}{a_1 + a_5 - \frac{a_5}{2}(a_1+a_2)} s^2 + (-1)^j - \frac{1}{2} \sqrt{D(s)} - \frac{1}{2} = \frac{(a_5 - a_2)s^2Y((-1)^j\epsilon)}{a_1(1-a_5s^2)} + \frac{(a_1 - a_5)Y((-1)^{j-1}\epsilon)}{a_1a_5}
\]
and
\[
\frac{a_5 - a_1}{2} + \frac{a_5(a_1 + a_2 - 2a_5)}{2} s^2 + (-1)^j \sqrt{D(s)} = \frac{(a_1 - a_5)(a_5s^2 - 1)Y((-1)^j\epsilon) + a_5(a_2 - a_5)s^2Y((-1)^{j-1}\epsilon)}{a_1a_5}
\]
and hence we obtain (with \( s_0 = \frac{1}{\sqrt{a_5}} \))
\[
E_2 = \frac{(a_1a_5)^{3/2}}{8\pi^2} Y(t) \begin{pmatrix} \frac{\partial_1 x_1}{|x|^2} & \frac{\partial_1 x_2}{|x|^2} & 0 \\ \frac{\partial_2 x_1}{|x|^2} & \frac{\partial_2 x_2}{|x|^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \int_{-s_0}^{s_0} \frac{2(t+x_3s)}{a_5(a_1 - a_2)s^2 + a_5 - a_1} \times
\]
\[
\times \left[ \frac{(a_5 - a_2)s^2}{a_1(1-a_5s^2)} Y(p_-) \frac{1}{\sqrt{p_-}} - \frac{a_1 - a_5}{a_1a_5} \frac{Y(p_+)}{\sqrt{p_+}} \right] ds
\]
\[
= \frac{Y(t)\sqrt{a_1a_5}}{4\pi^2} \begin{pmatrix} \frac{\partial_1 x_1}{|x|^2} & \frac{\partial_1 x_2}{|x|^2} & 0 \\ \frac{\partial_2 x_1}{|x|^2} & \frac{\partial_2 x_2}{|x|^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \times
\]
\[
\times \left\{ \int_{-s_0}^{s_0} (t+x_3s) \cdot \left[ \frac{1}{1-a_5s^2} + \frac{a_1 - a_5}{a_5(a_1 - a_2)s^2 + a_5 - a_1} \right] \frac{Y(p_-)}{\sqrt{p_-}} ds - \int_{-s_0}^{s_0} (t+x_3s) \cdot \frac{a_1 - a_5}{a_5(a_1 - a_2)s^2 + a_5 - a_1} \frac{Y(p_+)}{\sqrt{p_+}} ds \right\}.
\]
We deduce from the Lemma in 5.1 that

$$E_2 = \frac{Y(t)}{8\pi} \left( \begin{array}{ccc} \frac{x_1^2}{|x|^2} & \frac{x_2^2}{|x|^2} & 0 \\ \frac{x_1^2}{|x|^2} & \frac{x_2^2}{|x|^2} & 0 \\ 0 & 0 & 0 \end{array} \right) \left\{ \frac{|x|^2}{a_5} - t^2 \right\} \sum_{\pm} \frac{t \pm x_3/\sqrt{a_5}}{|t \pm x_3/\sqrt{a_5}|} - \sqrt{\frac{a_1 - a_5}{a_1 - a_2}} Y\left(\frac{|x|^2}{a_5} - t^2\right) \sum_{\pm} \frac{t \pm x_3\sqrt{\frac{a_1 - a_5}{a_5(a_1 - a_2)}}}{\sqrt{|x'|^2 \frac{a_1 - a_5}{a_5(a_1 - a_2)} + a_5(t \pm x_3\sqrt{\frac{a_1 - a_5}{a_5(a_1 - a_2)}})}^2} + \sqrt{\frac{a_1 - a_5}{a_5(a_1 - a_2)}} Y\left(\frac{|x'|^2}{a_1} + \frac{x_3^2}{a_2} - t^2\right) \sum_{\pm} \frac{t \pm x_3\sqrt{\frac{a_1 - a_5}{a_5(a_1 - a_2)}}}{\sqrt{|x'|^2 \frac{a_1(a_2 - a_5)}{a_5(a_1 - a_2)} + a_1(t \pm x_3\sqrt{\frac{a_1 - a_5}{a_5(a_1 - a_2)}})}^2}\right\}.$$ 

Employing, as in [38, p. 340], the notation

$$R_\pm = \sqrt{\left(\frac{a_1 - a_2}{a_5 x_3}\right)^2 + (a_2 - a_5)|x'|^2},$$

we finally obtain

$$E_2 = \frac{Y(t - \frac{|x|}{\sqrt{a_5}}) - Y(t - \frac{|x|}{\sqrt{a_5}})}{4\pi \sqrt{a_5} |x'|^4} \left( \begin{array}{ccc} x_2^2 - x_1^2 & -2x_1 x_2 & 0 \\ -2x_1 x_2 & x_1^2 - x_2^2 & 0 \\ 0 & 0 & 0 \end{array} \right) + \frac{\delta(t - \frac{|x|}{\sqrt{a_5}})}{4\pi a_5^{3/2} t |x'|^2} \left( \begin{array}{ccc} x_1^2 & x_1 x_2 & 0 \\ x_1 x_2 & x_2^2 & 0 \\ 0 & 0 & 0 \end{array} \right) + \frac{1}{8\pi \sqrt{a_5(a_1 - a_2)}} \left( \begin{array}{ccc} \frac{x_1}{|x|^2} & \frac{x_2}{|x|^2} & 0 \\ \frac{x_1}{|x|^2} & \frac{x_2}{|x|^2} & 0 \\ 0 & 0 & 0 \end{array} \right) \times \left[ Y\left(t - \frac{|x|}{\sqrt{a_5}}\right) - Y\left(t - \sqrt{\frac{|x'|^2}{a_1} + \frac{x_3^2}{a_2}}\right) \right] \cdot \partial_3(R_+ - R_-).$$

(Note that the unphysical term with the factor $Y(t - \frac{|x|}{\sqrt{a_5}})$ in $E_2$ is canceled by the corresponding term in $E_1$.)

If the remaining terms $E_3$ and $E_4$ are calculated similarly, we obtain

$$E_3 = -\frac{a_3}{8\pi \sqrt{a_5(a_1 - a_2)}} \left( \begin{array}{ccc} 0 & 0 & \partial_1 \\ 0 & 0 & \partial_2 \\ \partial_1 & \partial_2 & 0 \end{array} \right) \times \left[ Y\left(t - \frac{|x|}{\sqrt{a_5}}\right) - Y\left(t - \sqrt{\frac{|x'|^2}{a_1} + \frac{x_3^2}{a_2}}\right) \right] \cdot \left( \frac{1}{R_+} - \frac{1}{R_-} \right).$$
and

$$(E_4)_{33} = -\frac{(a_2 - a_5)\sqrt{a_1 - a_5}}{8\pi a_5(a_1 - a_2)} \partial_t \left[ Y(t - \frac{|x|}{\sqrt{a_5}}) - Y\left(t - \sqrt{\frac{|x'|^2}{a_1} + \frac{x_3^2}{a_2}}\right)\right] \times$$

$$\times \left(\frac{1}{R_+} - \frac{1}{R_-}\right) + \frac{(a_1 - a_5)\delta(t - \frac{|x|}{\sqrt{a_5}})}{4\pi a_5^{3/2}(a_1 - a_2)t} - \frac{(a_2 - a_5)\delta(t - \sqrt{\frac{|x'|^2}{a_1} + \frac{x_3^2}{a_2}})}{4\pi a_5\sqrt{a_2(a_1 - a_2)t}}.$$

Adding the four terms $E_1, E_2, E_3, E_4$ yields the fundamental matrix $E$ given explicitly in [38, Prop. 2, p. 339], cf. also [41, (3.10.54–58), pp. 103 f.] and [6, p. 669, Case (i)].

5.3. The case of $a_1 = a_5$, i.e. $c_{11} = c_{44}$

Here we shall show that the integrals in Prop. 5 are indeed elliptic ones if $a_1 = a_5$. These integrals are all of the form

$$J = \int R(s, \sqrt{D(s)}) \frac{ds}{\sqrt{\mu_j(s)}},$$

where

$$D(s) = a^2 s^2 (s^2 + b^2)$$

and $\mu_{1,2}(s) = ps^2 + qs + r + |x'|^2 \sqrt{D(s)}$

with (possibly complex) constants $a, b$, polynomials $p, q, r$ in $t, x$, and a rational function $R$. Performing the substitution

$$s = \frac{b}{2} \left(\xi - \frac{1}{\xi}\right)$$

yields

$$ds = \frac{b}{2} \left(1 + \frac{1}{\xi^2}\right), \sqrt{D(s)} = \frac{ab^2}{4} \left(\xi^2 - \frac{1}{\xi^2}\right), \mu_{1,2} = \frac{r_0 + r_1 \xi + r_2 \xi^2 + r_3 \xi^3 + r_4 \xi^4}{\xi^2}$$

(with polynomials $r_0, \ldots, r_4$ in $t, x$) and hence

$$J = \frac{b}{2} \int R\left(\frac{b}{2} \left(\xi - \frac{1}{\xi}\right), \frac{ab^2}{4} \left(\xi^2 - \frac{1}{\xi^2}\right)\right) \cdot \frac{\left(\xi + \frac{1}{\xi}\right)d\xi}{\sqrt{r_0 + r_1 \xi + r_2 \xi^2 + r_3 \xi^3 + r_4 \xi^4}},$$

which is an elliptic integral. Due to the factors $Y(\mu_j)$, the above integrals are indeed complete elliptic integrals.

In [42, 4., pp. 188–192], the special case of $a_1 = a_2 = a_5$ was considered and $E_{33}$ was represented by elliptic integrals, see also Remark 1 to Prop. 4.
6. The values of the fundamental matrix on symmetry planes

In the literature, the values of $E$ on the plane $x' = 0$ (cf. [41, Ch. 3, 11, pp. 105–111]) and on the hyperplane $x_3 = 0$ (cf. [8, 4., pp. 30–37]), respectively, have been calculated. These particular values can be deduced from our general formula in Prop. 5.

6.1. The values of $E$ on the hyperplane $x_3 = 0$

Here we shall show that $E(t,x_1,x_2,0)$ is expressible by complete elliptic integrals, whereas $E(t,x)$ in general is given by Abelian integrals of the genus 3, see Remark 1 to Prop. 5.

For $x_3 = 0$, $E_3$ vanishes and the integrals representing $E_2$ and $E_4$ in Prop. 5 are both of the form

$$J = \int R(s^2, \sqrt{D(s)}) \frac{ds}{\sqrt{\mu_j(s)}},$$

where

$$D(s) = a(s^2 + b)(s^2 + c) \quad \text{and} \quad \mu_{1,2}(s) = ps^2 + q \pm |x'|^2 \sqrt{D(s)}$$

with (possibly complex) constants $a, b, c$, polynomials $p, q$ in $t, x'$, and a rational function $R$. The substitution

$$\xi^2 = \frac{s^2 + c}{s^2 + b}$$

yields

$$s = \sqrt{\frac{b\xi^2 - c}{1 - \xi^2}}, \quad ds = \frac{(b - c)\xi d\xi}{(1 - \xi^2)^{3/2} \sqrt{b\xi^2 - c}}, \quad \sqrt{D(s)} = \frac{\sqrt{a(b - c)}\xi}{1 - \xi^2}, \quad \mu_{1,2} = \frac{r_0 + r_1 \xi + r_2 \xi^2}{1 - \xi^2}$$

(with polynomials $r_0, r_1, r_2$ in $t, x'$) and hence

$$J = \int \frac{(b - c)\xi}{1 - \xi^2}, R\left(\frac{b\xi^2 - c}{1 - \xi^2}, \frac{\sqrt{a(b - c)}\xi}{1 - \xi^2}\right), \frac{d\xi}{\sqrt{b\xi^2 - c}(r_0 + r_1 \xi + r_2 \xi^2)},$$

which is an elliptic integral. Due to the factors $Y(\mu_j)$, these integrals are indeed complete elliptic integrals.

In [8, (53), (54), p. 32], the solution of $P(\partial)u = Y(t)\delta(x)$, i.e. $u = E * Y(t)\delta(x)$ is represented by single definite integrals for $x_3 = 0$. These integrals correspond to $J$ after some trigonometric substitution and they are evaluated numerically in [8]. The fact that these integrals are elliptic went unnoticed in [8].

6.2. The values of $E$ on the plane $x_1 = x_2 = 0$

According to [40] (see also [41, Ch. 3, 11, pp. 105–111]), the values of $E$ on the symmetry axis $x_1 = x_2 = 0$ can be expressed by algebraic and delta functions. (“These results constitute the first (three-dimensional, time-dependent) extension of Stokes’ celebrated solution for an isotropic solid to a (physically realizable) anisotropic solid”, cf. [41, p. 105].) The following formula for $E(t,0,0,x_3)$ (see Prop. 6 below) could in principle be deduced from Prop. 5 by a limit process. Note that $\mu_{1,2}(s) = -a_1 a_5(t+x_3 s)^2 \leq 0$
if \( x' = 0 \), and hence the integration intervals for \( E_2, E_3, E_4 \) degenerate to the point \( s = -\frac{x}{t} \). The values of these integrals can then be deduced from the following limit:

\[
\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \phi(s) \frac{Y(\mu_e(s))}{\sqrt{\mu_e(s)}} \, ds = \frac{\pi Y(h(\frac{s}{\alpha})) \cdot \phi(\frac{\beta}{\alpha})}{\sqrt{A} \cdot |\alpha|},
\]

if \( A > 0, \alpha \in \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R}, \mu_e(s) = -A(\alpha s - \beta)^2 + \epsilon h(s) \)

with functions \( \phi, h \) which are continuous at \( \frac{\beta}{\alpha} \). It is easier to derive \( E(t,0,0,x_3) \) directly from the three-dimensional integral representation (8), i.e. from the Herglotz–Gårding formula.

**Proposition 6.** Let \( P(\partial), E, s_0, D(s) \) be as in Prop. 5 and set \( s_1 := \max(\frac{1}{\sqrt{a_2}}, \frac{1}{\sqrt{a_5}}) \) and \( s_2 := \min(\frac{1}{\sqrt{a_2}}, \frac{1}{\sqrt{a_5}}) \). Then the restriction of \( E \) to the plane \( x_1 = x_2 = 0 \) is given for \( t > 0 \) by

\[
E(t,0,0,x_3) = \frac{1}{16\pi|x_3|} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \left( \begin{pmatrix} 2 + \frac{1}{a_1} \end{pmatrix} \delta(t - \frac{|x_3|}{\sqrt{a_5}}) + \frac{1}{a_1} \delta(t - \frac{|x_3|}{\sqrt{a_2}}) \right) +
\]

\[
+ \frac{1}{32\pi a_1|x_3|} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \partial_t \left[ \chi(t |x_3|) a_1 a_5 + (a_1^2 - a_2 a_3) \frac{t^2}{x_3^3} \right] +
\]

\[
+ \frac{1}{8\pi a_5|x_3|} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left( \delta(t - \frac{|x_3|}{\sqrt{a_5}}) + \delta(t - \frac{|x_3|}{\sqrt{a_2}}) \right) +
\]

\[
+ \frac{1}{16\pi a_5 |x_3|} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial_t \left[ \chi(t |x_3|) a_5 - a_1 + (a_1 a_2 - a_2^2 - a_3^2) \frac{t^2}{x_3^3} \right],
\]

where \( \chi(s) := Y(s_0 - s) \cdot [Y(s - s_1) + Y(s - s_2)] \).

**Proof.** In order to calculate \( E(t,0,0,x_3) \), we proceed along the same steps as in the proof of Prop. 5. Denoting by \( E_1, E_2, E_3, E_4 \) the same summands of \( E \) as there, we obtain first (again with \( \rho^2 = \xi_1^2 + \xi_2^2 \)):

\[
E_1(t,0,0,x_3) = -\frac{Y(t)}{4\pi^2} \partial_t \int_{R^3} \frac{\xi T}{\rho^2} \delta(W_1(1,\xi)) \, d\xi \delta(t + x_3 \xi_3) \, d\xi
\]

\[
= -\frac{Y(t)}{4\pi^2|x_3|} \partial_t \int_{R^2} \frac{\xi T}{\rho^2} \delta(1 - a_4 \rho^2 - a_5 \frac{t^2}{x_3^3}) \, d\xi_1 d\xi_2
\]

\[
= -\frac{Y(t)}{8\pi a_4|x_3|} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \partial_t \left[ Y\left( \frac{|x_3|}{\sqrt{a_5}} - t \right) \right]
\]

\[
= -\frac{1}{8\pi a_4|x_3|} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \delta(t - \frac{|x_3|}{\sqrt{a_5}}),
\]
In a similar way, we can simplify $E_2$:

$$E_2(t, 0, 0, x_3) = -\frac{Y(t)\partial_t}{4\pi^2} \int_{\mathbb{R}^3} \frac{W_4(1, \xi)}{\rho^2} \begin{pmatrix} \frac{\xi_1^2}{\rho^2} & \xi_1 \xi_2 & 0 \\ \xi_1 \xi_2 & \frac{\xi_2^2}{\rho^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(R(1, \xi)) \sigma(\xi) \delta(t + x_3 \xi_3) \, d\xi$$

$$= -\frac{Y(t)}{4\pi |x_3|} \partial_t \int_{\mathbb{R}^2} \frac{1 - a_5 \rho^2 - a_2 \frac{t^2}{x_3^2}}{\rho^2} \begin{pmatrix} \frac{\xi_1^2}{\rho^2} & \xi_1 \xi_2 & 0 \\ \xi_1 \xi_2 & \frac{\xi_2^2}{\rho^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \times$$

$$\times \delta(R(1, \xi_1, \xi_2, \frac{t}{x_3})) \sigma(\xi_1, \xi_2, \frac{t}{x_3}) \, d\xi_1 d\xi_2$$

$$= -\frac{Y(t)}{4\pi |x_3|} \left( I_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \partial_t \int_0^\infty \left( 1 - a_5 \rho^2 - a_2 \frac{t^2}{x_3^2} \right) \delta(R(1, \rho, 0, \frac{t}{x_3})) \sigma(\rho, 0, \frac{t}{x_3}) \, d\rho \right).$$

By means of formula (12) in the proof of Prop. 5, using the sign factors $\epsilon_j$ from Prop. 5 and setting

$$\lambda_{1,2}(s) = \frac{1}{a_1 a_5} \left( \frac{a_1 + a_5}{2} - a_6 s^2 \pm \sqrt{D(s)} \right), \quad \lambda_1(s) \geq \lambda_2(s),$$

we obtain

$$E_2(t, 0, 0, x_3) = -\frac{Y(t)}{16\pi |x_3|} \left( I_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \partial_t \left[ \frac{Y(s_0 - \frac{t}{|x_3|})}{\sqrt{D(\frac{t}{x_3})}} \times \right.ight.$$

$$\left. \left. \times \sum_{j=1}^2 \epsilon_j \left( 1 - a_5 \lambda_j(\frac{t}{x_3}) - a_2 \frac{t^2}{x_3^2} \right) Y(\lambda_j(\frac{t}{x_3})) \right] \right).$$

Since the smaller root $\lambda_2(s)$ of $\lambda \mapsto R(1, \sqrt{\lambda}, 0, s)$ becomes negative for $s_2 \leq s \leq s_1$ and since $(t, 0, 0, x_3) \in B$ iff $s_1 \leq \frac{t}{|x_3|} \leq s_0$, we obtain

$$\frac{Y(t)Y(s_0 - \frac{t}{|x_3|})}{\sqrt{D(\frac{t}{x_3})}} \sum_{j=1}^2 \epsilon_j \left( 1 - a_5 \lambda_j(\frac{t}{x_3}) - a_2 \frac{t^2}{x_3^2} \right) Y(\lambda_j(\frac{t}{x_3})) =$$

$$= -\chi \left( \frac{t}{|x_3|} \right) \frac{a_1 - a_5 + (a_5^2 - a_1 a_2 - a_3^2) \frac{t^2}{x_3^2}}{2a_1 \sqrt{D(\frac{t}{x_3})}} + \frac{Y(t)}{a_1} \sum_{j=1}^2 Y(s_j - \frac{t}{|x_3|}).$$

Hence

$$E_2(t, 0, 0, x_3) = \frac{1}{32 \pi a_1 |x_3|} \left( I_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \partial_t \left[ \chi \left( \frac{t}{|x_3|} \right) \frac{a_1 - a_5 + (a_5^2 - a_1 a_2 - a_3^2) \frac{t^2}{x_3^2}}{\sqrt{D(\frac{t}{x_3})}} \right] \right.$$}

$$\left. + \frac{1}{16 \pi a_1 |x_3|} \left( I_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \left( \delta \left( t - \frac{|x_3|}{\sqrt{a_2}} \right) + \delta \left( t - \frac{|x_3|}{\sqrt{a_5}} \right) \right) \right).$$
Next we observe that
\[
E_3(t, 0, 0, x_3) = -\frac{a_3 Y(t)}{4\pi^2} \partial_t \int_{\mathbb{R}^3} \xi_3 \begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix} \delta(R(1, \xi)) \sigma(\xi) \delta(t + x_3 \xi_3) \, d\xi
\]
vansishes, since the integrand is an odd function with respect to \( \xi_1, \xi_2 \).

Finally, we calculate \( E_4(t, 0, 0, x_3) \) in exactly the same way as \( E_2(t, 0, 0, x_3) \):\[
E_4(t, 0, 0, x_3)_{33} = -\frac{Y(t)}{4\pi^2} \partial_t \int_{\mathbb{R}^3} W_3(1, \xi) \delta(R(1, \xi)) \sigma(\xi) \delta(t + x_3 \xi_3) \, d\xi
\]
\[
= -\frac{Y(t)}{2\pi^2 |x_3|} \partial_t \int_{\mathbb{R}^2} (1 - a_1 \rho^2 - a_5 \frac{t^2}{x_3^2}) \delta(R(1, \xi_1, \xi_2, \frac{t}{x_3})) \sigma(\xi_1, \xi_2, \frac{t}{x_3}) \, d\xi_1 \, d\xi_2
\]
\[
= -\frac{Y(t)}{8\pi |x_3|} \partial_t \left[ \frac{Y(s_0 - \frac{t}{|x_3|})}{\sqrt{D(\frac{t}{x_3})}} \sum_{j=1}^2 \epsilon_j (1 - a_1 \lambda_j(\frac{t}{x_3}) - a_5 \frac{t^2}{x_3^2}) Y(\lambda_j(\frac{t}{x_3})) \right].
\]

We then conclude the calculation employing the following identity, which is completely analogous to (13):
\[
\frac{Y(t) Y(s_0 - \frac{t}{|x_3|})}{\sqrt{D(\frac{t}{x_3})}} \sum_{j=1}^2 \epsilon_j (1 - a_1 \lambda_j(\frac{t}{x_3}) - a_5 \frac{t^2}{x_3^2}) Y(\lambda_j(\frac{t}{x_3})) =
\]
\[
= -\chi(\frac{t}{|x_3|}) \frac{a_5 - a_1 + (a_1 a_2 - a_5^2 - a_3^2) \frac{t^2}{x_3^2}}{2a_5 \sqrt{D(\frac{t}{x_3})}} + \frac{Y(t)}{a_5} \sum_{j=1}^2 Y(s_j - \frac{t}{|x_3|}). \tag*{\square}
\]

**Remark.** The result of Prop. 6 is in complete agreement with the formulas in [40, (4.1)–(4.16), pp. 77, 78] and [41, (3.11.22)–(3.11.37), p. 109] if the following correspondence of notations is taken into account: \( \alpha = \frac{a_2}{a_5}, \beta = \frac{a_1}{a_5}, \tau = x_3, \bar{\tau} = \frac{\tau}{\tau}, u_1 = a_5 E_{11} \ast (Y(t) \otimes \delta(x)), w_3 = a_5 E_{33} \ast (Y(t) \otimes \delta(x)) \).

**Appendix**

We use the following five elastic constants \( a_1, \ldots, a_5 \) determining a transversely isotropic medium, and, at some instances, the additional quantities \( a_6, a_7 \) :
\[
a_1 = c_{1111} = c_{11}, \quad a_2 = c_{3333} = c_{33}, \quad a_3 = c_{1133} + c_{2323} = c_{13} + c_{44},
\]
\[
a_4 = \frac{1}{2} (c_{1111} - c_{1122}) = \frac{1}{2} (c_{11} - c_{12}), \quad a_5 = c_{2323} = c_{44},
\]
\[
a_6 = \frac{1}{2} (a_1 a_2 + a_2^2 - a_3^2), \quad a_7 = (a_5^2 - a_1 a_2)(a_1 - a_5) + a_3^2(a_1 + a_5).
\]  \tag{2}

Thereby the wave operators \( W_1, \ldots, W_5 \) are determined as follows:
\[
W_1(\partial) = \partial_i^2 - a_1 \Delta_2 - a_5 \partial_3^2, \quad W_2(\partial) = (a_1 - a_4) W_4(\partial) + a_3^2 \partial_3^2,
\]
\[
W_3(\partial) = \partial_i^2 - a_1 \Delta_2 - a_5 \partial_3^2, \quad W_4(\partial) = \partial_i^2 - a_5 \Delta_2 - a_2 \partial_3^2,
\]
\[
W_5(\partial) = \partial_i^2 - a_1 \Delta_2 - \frac{a_1 + a_2}{2} \partial_3^2.
\]
REFERENCES


