ZEILON'S OPERATOR AND LACUNAE

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First, some glimpses at history are taken. Mainly we examine the connexion of Zeilon's operator $\partial_1^3 + \partial_2^3 + \partial_3^3$ with the investigation of lacunae of fundamental solutions of linear partial differential operators with constant coefficients. Second, some essential results from the work of Petrovsky, Atiyah, Bott, and Gårding on the lacunae of fundamental solutions of hyperbolic operators are repeated and illustrated by examples. Third, we present a slight generalization of that work to non-hyperbolic operators and we sketch its application to Zeilon's operator.

In this introductory paper, I try to maintain the informal (and hopefully lively) style appropriate to what had originally been prepared for a lecture at the PDE-Workshop in February 2001 at Graz. For more mathematical details, I refer to [16] and [17].

1 Historical Remarks

1) Ivar Fredholm (1866–1927) became famous by his foundation of the theory of integral equations. Less known are his four articles (of only 18 in all; so quality and quantity differ at times) on the calculation of fundamental solutions. One of those 4 items was his thesis, which also was his longest paper (42 pages), and which was published in Acta Mathematica in 1900 ([4]). Fredholm starts with the sentence "On connait le rôle fondamental que joue l'intégrale particulière $\frac{1}{r}$ de l'équation $\Delta u = 0$ dans la théorie du potentiel." and then sets out to construct fundamental matrices F of elliptic 3 by 3 systems $A(\partial)$ of linear partial differential operators in three variables with constant coefficients and homogeneous of second order. (We notice, in parentheses, that the notion of fundamental solution could be defined in general only after the advent of L. Schwartz' theory of distributions (cf. [15]), and therefore Fredholm made reference to the function $\frac{1}{r}$, which is, up to the constant factor -4π, a fundamental solution of Δ in $\mathbf{R}^3$.) The respective formula (10) on p. 7 in Fredholm's elegant paper has, in modern language, the following appearance (cf. [13, 3.2.2, (F′)]):

$$F(x) = -\frac{\mathrm{i}\,\mathrm{sign}\,(x_2)}{2\pi} \sum_{k=1}^{3} \frac{|\zeta_k(x)|^2\, A(\zeta_k(x))^{\mathrm{ad}}}{\det\bigl(x, \overline{\zeta_k(x)}, (\nabla \det A)(\zeta_k(x))\bigr)},$$

where $A(\zeta)^{\mathrm{ad}}$ denotes the adjoint matrix of $A(\zeta)$ and $\zeta_k(x) \in \mathbf{C}^3 \setminus \{0\}$ are determined up to complex factors by the conditions

$$\langle \zeta_k(x), x \rangle = 0, \quad \det A(\zeta_k(x)) = 0, \quad \mathrm{Im}\,\Bigl(\frac{\zeta_k(x)_1}{\zeta_k(x)_3}\Bigr) > 0, \quad k = 1, 2, 3.$$

(Here and in the following, $\langle\ ,\ \rangle$ denotes the inner product on $\mathbf{R}^n$ or $\mathbf{C}^n$, respectively.)

In his last paper on fundamental solutions [5], Fredholm calculated a fundamental solution E (actually the only one which is homogeneous and even) of the operator $\partial_1^4 + \partial_2^4 + \partial_3^4$ and obtained (up to the constant factor $-\frac{1}{8\pi}$) the following formula:

$$\begin{aligned} E(x) &= -\frac{1}{8\pi} \sum_{j=1}^{3} |x_j| \int_{t/(2x_j^2)}^{\infty} \frac{\mathrm{d}u}{\sqrt{4u^3 - u}} \\ &= -\frac{1}{8\pi} \sum_{j=1}^{3} x_j F\Bigl(\arcsin\Bigl(\frac{\sqrt{2}\, x_j}{\sqrt{t + x_j^2}}\Bigr), \frac{1}{\sqrt{2}}\Bigr). \end{aligned}$$

Therein t denotes the largest of the three real roots of the cubic

$$t^3 - (x_1^4 + x_2^4 + x_3^4)t - 2x_1^2 x_2^2 x_3^2 = 0,$$

and F denotes the elliptic integral of the first kind (cf. [7, 3.131.8 and 8.111]).
2) Niels Zeilon (1886–1957) was a student and admirer of Fredholm (see [6]). He generalized Fredholm's method of construction of fundamental matrices to arbitrary real homogeneous systems with the aim of describing the propagation of light in biaxial crystals. Before reaching this ambitious goal in 1919–1921 through his two big papers (cf. [19]), he considered in [18], presumably stimulated by Fredholm's last-mentioned work, the operator $\partial_1^3 + \partial_2^3 + \partial_3^3$. I therefore designated this operator as "Zeilon's operator". Zeilon did not obtain an explicit representation for a fundamental solution as Fredholm did in his case, but he detected that the fundamental solution E which he considers becomes singular only along the surfaces $\partial L \cup -\partial L$, where ∂L denotes the boundary of the cone

$$\begin{aligned} L := \{x \in \mathbf{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0,& \\ x_i^{3/2} < x_j^{3/2} + x_k^{3/2} \text{ for all permutations } i, j, k \text{ of } 1, 2, 3\}.& \end{aligned}$$

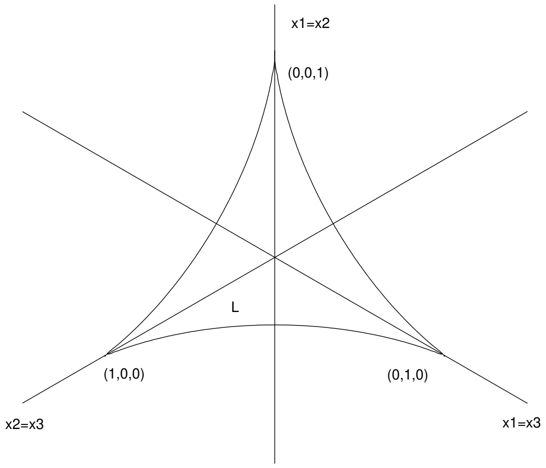

Figure 1. The intersection of L with the plane $x_1 + x_2 + x_3 = 1$

3) Thereafter Gustav Herglotz, Ivan Petrovsky, and Jean Leray gave representations for the (retarded) fundamental solutions of homogeneous hyperbolic operators with constant coefficients (cf. [8], [14], [10]). They also investigated conditions for the appearance of lacunae. Their theory was subsumed and generalized in 1970/73 by Michael Atiyah, Raoul Bott and Lars Gårding in the seminal papers [1]. Building up on them Johan Fehrman introduced in 1975 the class of hybrid operators, which, by definition, possess fundamental solutions that are real-analytic outside proper cones. As an example, he showed that $\partial_1^3 + \partial_2^3 + \partial_3^3$ is hybrid with respect to the direction $N = (1, 1, 1)$ (see [2, p. 223]) and, therefore, it possesses a fundamental solution which is real analytic outside the wave front surface ∂L with respect to N (see [2, Th. 4, p. 231]). He also proved that this fundamental solution of $\partial_1^3 + \partial_2^3 + \partial_3^3$ has (except at the origin) sharp fronts everywhere from within L (see [2, p. 235]). However, he did not give an explicit formula for a fundamental solution exhibiting this behaviour.

4) In 1990, Reinhold Meise, B. Alan Taylor, and Dietmar Vogt gave conditions for the existence of a "right inverse" R of a linear partial differential operator $P(\partial)$ with constant coefficients ([11]). In particular, they showed that there exists a right inverse on $\mathbf{R}^n$, i.e. a continuous linear operator $R : \mathcal{D}'(\mathbf{R}^n) \longrightarrow$

$\mathcal{D}'(\mathbf{R}^n)$ satisfying $P(\partial)\circ R=\mathrm{id}$, if and only if

$$\forall r>0:\exists\rho>0:\forall x_0\in\mathbf{R}^n \text{ with } |x_0|\geq\rho:\exists E\in\mathcal{D}'(\mathbf{R}^n): \\ P(\partial)E=\delta \text{ and } E|_{\{x\in\mathbf{R}^n:|x-x_0|<r\}}=0,$$

or, put more vaguely, iff $P(\partial)$ has enough fundamental solutions with large lacunae. Zeilon's operator $\partial_1^3+\partial_2^3+\partial_3^3$ has this property (cf. [11, p. 651]), but it is no evolution operator and hence it admits no right inverse on bounded convex sets Ω instead of $\mathbf{R}^3$ (cf. [12, Ex. 1, p. 463], [3, Ex. 3.7, p. 160]).

In the paper [17], I explicitly calculated a fundamental solution E of Zeilon's operator. It results that E is constant in L and in $-L$ and hence there exist fundamental solutions of $P(\partial)$ possessing these cones as lacunae. It is still an open question to decide whether there exist fundamental solutions of $P(\partial)$ having *conical* lacunae different from L and $-L$.

More in detail, I showed in [17] that Zeilon's fundamental solution E has in $\pm L$ the constant values $\mp B(\frac13,\frac13)/(8\sqrt{3}\,\pi)$ (where B denotes Euler's beta function), and that, outside $\pm L$, it is given by the formula

$$\begin{aligned}E(x)&=\frac{\sqrt[3]{2}}{8\sqrt{3}\,\pi}\operatorname{sign}(3\sqrt[3]{2}\,x_1x_2x_3-x_1^3-x_2^3-x_3^3)\int_{-1}^{t}\frac{\mathrm{d}u}{\sqrt{u^3+1}}\\&=\frac{\sqrt[3]{2}}{8\cdot 3^{3/4}\,\pi}\operatorname{sign}(3\sqrt[3]{2}\,x_1x_2x_3-x_1^3-x_2^3-x_3^3)\\&\qquad\times F\Big(\arccos\Big(\frac{\sqrt{3}-1-t}{\sqrt{3}+1+t}\Big),\frac{\sqrt{3}+1}{2\sqrt{2}}\Big),\end{aligned}$$

where t is the only simple real root or, if x lies on one of the co-ordinate axes, t is the triple root 0, respectively, of the cubic equation

$$(x_1^6+x_2^6+x_3^6-2x_1^3x_2^3-2x_1^3x_3^3-2x_2^3x_3^3)t^3-9\sqrt[3]{4}\,x_1^2x_2^2x_3^2t^2 \\ -3\sqrt[3]{16}\,x_1x_2x_3(x_1^3+x_2^3+x_3^3)t-4(x_1^3x_2^3+x_1^3x_3^3+x_2^3x_3^3)=0.$$

In particular, this implies that the level sets of E are algebraic, and the same also holds for arbitrary real homogeneous cubic operators in three variables, as follows from the results in [16]. For quartic operators, a systematic analysis comparable to [16] for cubic ones is still missing.

2 Hyperbolic operators (according to Atiyah, Bott & Gårding)

In the sequel, we assume $P(\xi)=\sum_{\substack{\alpha\in\mathbf{N}_0^n\\|\alpha|\leq m}}a_\alpha\xi^\alpha$, $m=\deg P$, and $\vartheta\in\mathbf{R}^n\setminus\{0\}$.
We set $P_{\mathrm{pr}}(\xi):=\sum_{\substack{\alpha\in\mathbf{N}_0^n\\|\alpha|=m}}a_\alpha\xi^\alpha$.

Definition. The polynomial $P(\xi)$ (or the operator $P(-\mathrm{i}\,\partial)$ respectively) is called hyperbolic with respect to ϑ, iff

$$\text{(i) } P_{\mathrm{pr}}(\vartheta) \neq 0 \text{ and (ii) } \exists t_0 \in \mathbf{R} : \forall t < t_0 : \forall \xi \in \mathbf{R}^n : P(\xi + \mathrm{i}\, t\vartheta) \neq 0.$$

Lemma. Suppose that P is homogeneous and that $P(\vartheta) \neq 0$. Then the following conditions are equivalent:
(a) P is hyperbolic with respect to ϑ,
(b) $\forall \xi \in \mathbf{R}^n :$ the polynomial $t \longmapsto P(\xi + t\vartheta)$ has real roots only.
In particular, if (a) and (b) hold, then P is—up to a constant factor—a real polynomial.

PROOF. (a) implies (b): If $P(\xi+s\vartheta) = 0$ for $s \in \mathbf{C}$, then, by the homogeneity of P, we have $P(\sigma\xi + \sigma s\vartheta) = 0$ for all $\sigma \in \mathbf{R}$. From condition (ii) in the definition of hyperbolicity, we then deduce that $\mathrm{Im}(\sigma s) = \sigma \mathrm{Im}\, s \geq t_0$ for all $\sigma \in \mathbf{R}$ and hence $\mathrm{Im}\, s = 0$. Thus (b) above is fulfilled.
(b) implies (a): If $t \longmapsto P(\xi + t\vartheta)$ has only real roots, then, in particular, $P(\xi + \mathrm{i}\, t\vartheta) \neq 0$ for $t \in \mathbf{R} \setminus \{0\}$ and therefore condition (ii) in the definition of hyperbolicity is satisfied upon setting $t_0 = 0$.

In the sequel, we assume that P is homogeneous of the degree m and that $P(\vartheta) \neq 0$. We denote by $\mathbf{RP}^{n-1}$ real projective space, i.e. $\mathbf{RP}^{n-1} := \{[\xi] : \xi \in \mathbf{R}^n \setminus \{0\}\}$, where $[\xi] := \{c\xi : c \in \mathbf{R}\}$. (Analogously, complex projective space $\mathbf{CP}^{n-1}$ is defined.) Then $X := \{[\xi] \in \mathbf{RP}^{n-1} : P(\xi) = 0\}$ is the real projective algebraic variety defined by P. Furthermore, we denote by Γ that connectivity component of the set $\{\xi \in \mathbf{R}^n : P(\xi) \neq 0\}$ which contains ϑ.

Visualization of hyperbolicity in projective space.

If ξ, ϑ are linearly independent, then they span a line in projective space, namely $g_\xi := \{[\eta] : \xi, \vartheta, \eta \text{ linearly dependent}\}$. By the preceding lemma, P is hyperbolic with respect to ϑ iff, for all ξ linearly independent of ϑ, g_ξ has (counted with multiplicity) exactly m projective intersection points with X. In fact, due to $P(\vartheta) \neq 0$, the polynomial $P(\xi + t\vartheta)$ is of the degree m in t. If $P(\xi + t_j\vartheta) = 0$, $j = 1, \dots, m$, are its zeros (counted with multiplicity), which by (b) of the Lemma are all real, then $[\xi + t_j\vartheta]$, $j = 1, \dots, m$, are the intersection points of g_ξ with X, and vice versa.

EXAMPLES. 1) Let $P(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2$. Then $P(\partial)$ is the wave operator in two space dimensions. X is a circle in this case, and Figure 2 below (where $\xi_1 = 0$ represents the line at infinity) shows that P is hyperbolic with respect to ϑ iff $\vartheta_1^2 > \vartheta_2^2 + \vartheta_3^2$.

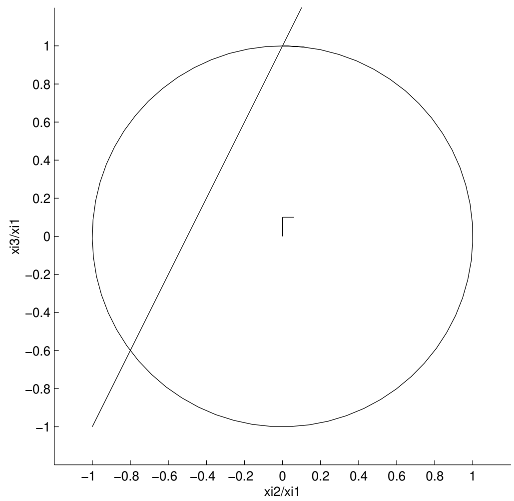

Figure 2. Hyperbolicity of the wave operator

2) In the case of $n = m = 3$, X is an elliptic curve. In Hesse's normal form, X has the equation $P(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3 = 0$. Figure 3 below, where $\xi_3 = 1$, shows that P can be hyperbolic with respect to some ϑ only if $a < -1$. In particular, Zeilon's operator is *not* hyperbolic with respect to any ϑ.

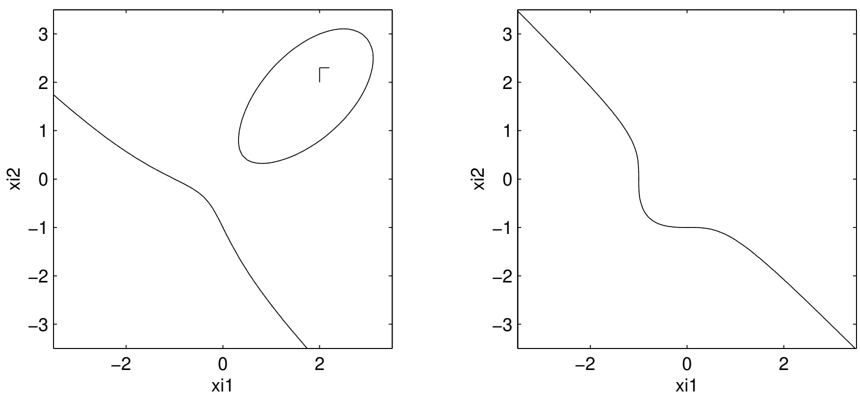

Figure 3. Real elliptic curves; left: $a < -1$, right: $a > -1$

We assume that P is homogeneous of the degree m and hyperbolic with respect to ϑ.

(i) $\exists ! E = E(P) \in \mathcal{D}'(\mathbf{R}^n) : P(\partial)E = \delta$ and $\operatorname{supp} E \subset \{x \in \mathbf{R}^n : \langle \vartheta, x\rangle \geq 0\}$;
(ii) $K = K(P) :=$ convex hull $(\operatorname{supp} E) = \Gamma^{\text{dual}} := \{x \in \mathbf{R}^n : \forall \xi \in \Gamma : \langle \xi, x\rangle \geq 0\}$;
(iii) $\bigcup_{\xi \in \mathbf{R}^n \setminus \{0\}} \operatorname{supp} E(P_\xi) \subset \operatorname{sing\,supp}_{(\mathrm{A})} E(P) \subset W := \bigcup_{\xi \in \mathbf{R}^n \setminus \{0\}} K(P_\xi)$,

where P_ξ is the localization of P at ξ, i.e.,

$$P(\xi + \eta) = P_\xi(\eta) + o\big(|\eta|^{\deg P_\xi}\big) \text{ for } \eta \to 0.$$

(P_ξ is likewise hyperbolic with respect to ϑ.)

(iv) If $|\nu| > m - n$ and $x \in K \setminus W$, then $\partial^\nu E(x) = \langle \widetilde{\alpha_x}, \widetilde{\omega_{x,\nu}}\rangle = \int_{\alpha_x} \omega_{x,\nu}$,

where (a) $\widetilde{\alpha_x}$ belongs to the homology group $H_{n-2}(U_x)$, which is the factor group of cycles modulo boundaries on the space

$$U_x := \{[\zeta] \in \mathbf{CP}^{n-1} : \langle \zeta, x\rangle = 0, P(\zeta) \neq 0\}.$$

Explicitly, α_x is given by the map

$$\alpha_x : S_x \longrightarrow \mathbf{CP}^{n-1} : \xi \longmapsto [\xi - \mathrm{i}\, v(\xi)], \qquad S_x := \{\xi \in \mathbf{R}^n : |\xi| = 1, \xi \perp x\},$$

v being a $\mathcal{C}^\infty$ function on S_x which fulfills

$$\forall \xi \in S_x : v(-\xi) = v(\xi),\ v(\xi) \in \Gamma(P_\xi),\ v(\xi) \perp \xi,\ \forall 0 < t \leq 1 : P\big(\xi - \mathrm{i}\, t v(\xi)\big) \neq 0.$$

(b) $\widetilde{\omega_{x,\nu}}$ belongs to de Rham's cohomology group $H^{n-2}(U_x)$, which is the factor group of closed $(n-2)$–forms modulo exact forms on U_x. Explicitly, $\omega_{x,\nu}$ is given by

$$\omega_{x,\nu} = (2\pi\,\mathrm{i})^{1-n} \Big[\Big(-\frac{x}{|x|^2}\, \nabla_\zeta\Big)^{|\nu|-(m-n)-1} \Big(\frac{\zeta^\nu}{P(\zeta)}\Big) \eta_x(\zeta) \Big],$$

where η_x is the "Leray form" on $V_x := \{\zeta \in \mathbf{C}^n : \zeta \perp x\}$, i.e., $\eta_x = \lambda_x|_{V_x}$ and

$$\mathrm{d}(\langle \zeta, x\rangle) \wedge \lambda_x(\zeta) = \sum_{j=1}^n (-1)^{j-1} \zeta_j \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_{j-1} \wedge \mathrm{d}\zeta_{j+1} \wedge \ldots \wedge \mathrm{d}\zeta_n + O(\langle \zeta, x\rangle)$$

for $\langle \zeta, x\rangle \to 0$, the equation holding in the space $\Omega^{n-1}(\mathbf{C}^n)$ of $(n-1)$–forms on $\mathbf{C}^n$. We also use η_x to orient the $(n-2)$–dimensional sphere S_x on which α_x is defined.

(v) If $x \in K \setminus W$ and $\widetilde{\alpha_x} = 0$ (the Petrovsky condition), then E is a polynomial of the degree $m-n$ in the connectivity component of x in $K \setminus W$. (The respective components of $K \setminus W$ are called "Petrovsky lacunae" of E.)

EXAMPLES. 3) Let $n = 3$, $P(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2$, $\vartheta = (1,0,0)$, and $E = E(P)$ as in (i) above. Then $\Gamma = \{\xi \in \mathbf{R}^3 : \xi_1 > |\xi'|\}$, where $\xi' = (\xi_2, \xi_3)$. The dual cone of Γ is $K = \{x \in \mathbf{R}^3 : x_1 \geq |x'|\}$ (here $x' = (x_2, x_3)$) and, by (ii), K contains $\operatorname{supp} E$.

For $\xi \in \mathbf{R}^3$ with $P(\xi) \neq 0$, the polynomial P_ξ equals the constant $P(\xi)$ and hence $E(P_\xi)$ is a constant times the delta function. If $P(\xi) = 0$ and $\xi \neq 0$, then

$$P_\xi(\eta) = 2\xi_1\eta_1 - 2\xi_2\eta_2 - 2\xi_3\eta_3$$

is a linear polynomial and the fundamental solution $E(P_\xi)$ has as support a half-ray on the forward light cone. Hence (iii) yields $\operatorname{sing\,supp} E = W = \{x \in \mathbf{R}^3 : x_1 = |x'|\}$.

Finally, let us apply the formula in (iv).
The homogeneity and the Lorentz invariance of the operator carry over to E, and we thus have $E = cY(x_1 - |x'|)/\sqrt{x_1^2 - |x'|^2}$. ($Y$ denotes Heaviside's function.) It is thus sufficient to calculate $c = E(x)$ for $x = (1,0,0)$. Then $S_x = \{(0,\xi') \in \mathbf{R}^3 : |\xi'| = 1\}$. For $\xi \in S_x$, $P_\xi(\eta)$ is the constant polynomial $P(\xi) = -1$ and $\Gamma(P_\xi) = \mathbf{R}^3$. Therefore we can set v equal to zero. Moreover, $U_x = \{[0, \zeta_2, \zeta_3] \in \mathbf{CP}^2 : \zeta_2^2 + \zeta_3^2 \neq 0\}$ is homeomorphic to $(\mathbf{C} \setminus \{\pm \mathrm{i}\}) \cup \{\infty\}$ upon using the parametrization $z = \zeta_2/\zeta_3$. Since $\eta_x(\zeta) = \zeta_3 \mathrm{d}\zeta_2 - \zeta_2 \mathrm{d}\zeta_3$ furnishes the orientation on S_x, the circle S_x in the $[\xi_2, \xi_3]-$plane is run through clockwise. Hence, in the variable z, α_x corresponds to twice the real line oriented in the usual sense. For $\zeta = (0, \zeta_2, \zeta_3)$, we have

$$\frac{\eta_x(\zeta)}{P(\zeta)} = \frac{\zeta_3 \mathrm{d}\zeta_2 - \zeta_2 \mathrm{d}\zeta_3}{-\zeta_2^2 - \zeta_3^2} = -\frac{\mathrm{d}z}{1+z^2}$$

and we conclude that

$$c = E(1,0,0) = E(x) = \frac{1}{(2\pi\,\mathrm{i})^2} \int_{\alpha_x} \left[\frac{\eta_x(\zeta)}{P(\zeta)}\right] = -\frac{2}{4\pi^2} \int_{\mathbf{R}} \frac{-\mathrm{d}z}{1+z^2} = \frac{1}{2\pi}.$$

4) Let us consider the wave operator in $\mathbf{R}^4$, i.e., $n = 4$, $P(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2$, $\vartheta = (1,0,0,0)$, and $E = E(P)$. As in Ex. 3, $K = \{x \in \mathbf{R}^4 : x_1 \geq |x'|\}$ and $\operatorname{sing\,supp} E = W = \{x \in \mathbf{R}^4 : x_1 = |x'|\}$, where now $x' = (x_2, x_3, x_4)$. Let us check that the Petrovsky condition is satisfied for $x = (1,0,0,0)$.

In this case, S_x is the unit sphere in the $[\xi_2, \xi_3, \xi_4]-$space oriented by

$$\eta_x(\xi) = -\xi_2 \mathrm{d}\xi_3 \wedge \mathrm{d}\xi_4 + \xi_3 \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_4 - \xi_4 \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3.$$

Again, v can be chosen as 0. Let us identify U_x with $\{[\zeta_2, \zeta_3, \zeta_4] \in \mathbf{CP}^2 : \zeta_2^2 + \zeta_3^2 + \zeta_4^2 \neq 0\}$. Then, with α_x, we twice cover the real projective space $\mathbf{RP}^2$ contained in U_x. Since $\eta_x(-\xi) = -\eta_x(\xi)$, the homology class $\widetilde{\alpha_x}$ vanishes and, by (v) of the catalogue above, we conclude that $E = 0$ in $K \setminus W$. In fact, as is well known, $E = \delta(x_1 - |x'|)/(4\pi x_1)$ has such a Petrovsky lacuna inside the forward light cone.
5) Finally, let us consider $n = m = 3$ and $P(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3$ (cf. Ex. 2 above) for $a < -1$. Then P is hyperbolic with respect to $\vartheta = (1, 1, 1)$. By [16, p. 284], $W = \{x \in \mathbf{R}^3 : A(x) = 0,\ x_1 + x_2 + x_3 \geq 0\}$, where

$$A(x) := 3a(a^3+4)x_1^2x_2^2x_3^2 + 4(a^3+1)(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) \\ +6a^2x_1x_2x_3(x_1^3 + x_2^3 + x_3^3) - (x_1^3 + x_2^3 + x_3^3)^2.$$

The intersection of W with the plane $x_1 + x_2 + x_3 = 1$ is depicted in Figure 4 below.

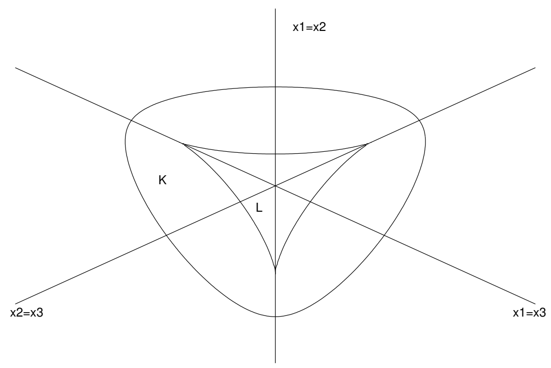

Figure 4. The wave surface W for cubic hyperbolic operators

The propagation cone K is the convex hull of the outer, convex part of W. Let us choose x inside the inner, non-convex cone and y in between the two conical surfaces which make up W. U_x is the Riemannian sphere $\{[\zeta] \in \mathbf{CP}^2 : \langle \zeta, x \rangle = 0\}$ with three points, the zeros of $P(\zeta)$, excluded. All these zeros belong to the real axis $\{[\xi] \in \mathbf{CP}^2 : \langle \xi, x \rangle = 0, \xi \in \mathbf{R}^3 \setminus \{0\}\}$ in U_x. The cycle α_x twice covers the real axis turning around the real zeros of P once on both sides (cf. Fig. 5, left part). Therefore α_x can be contracted in U_x and thus is homologous to 0, i.e., $[\alpha_x] = 0$. Hence E has a Petrovsky lacuna inside the non-convex part L of W.

For y however, one zero belongs to the real axis in U_y and two complex conjugate zeros lie off the real axis. (cf. Fig. 5, right part). Hence the homology class of α_y does not vanish in $H_1(U_y)$.

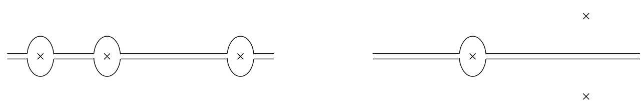

Figure 5. The cycles α_x (left) and α_y (right)

3 Generalization to non-hyperbolic operators

Although Zeilon's operator $\partial_1^3 + \partial_2^3 + \partial_3^3$ is not hyperbolic (cf. II, Ex. 2), the fundamental solution considered by N. Zeilon nevertheless has a lacuna (in the sense of being constant) inside the cone L depicted in Fig. 1. This is implied by the following proposition, which carries over part of the statements of Atiyah, Bott, and Gårding to the case where *the dimension n is odd and P is of principal type*, i.e., $\nabla P_{\mathrm{pr}}(\xi) \neq 0$, $\forall \xi \in \mathbf{R}^n \setminus \{0\}$ (cf. [9, II, Def. 10.4.11, p. 38]). As above, Y denotes Heaviside's function. Moreover, we define the Fourier transform as the extension by duality (or by continuity, which comes up to the same thing) of $\mathcal{F} : \mathcal{S}(\mathbf{R}^n) \longrightarrow \mathcal{S}(\mathbf{R}^n) : \phi \longmapsto \int_{\mathbf{R}^n} \phi(\xi) \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle}\,\mathrm{d}\xi$ to the space $\mathcal{S}'(\mathbf{R}^n)$ of tempered distributions. Pf denotes the finite part of a meromorphic distribution-valued function. For more details, see [16, 2., p. 288].

Proposition. Suppose that n is odd and that P is a real polynomial in n variables, of principal type and homogeneous of the degree m. Define $E \in \mathcal{S}'(\mathbf{R}^n)$ by

$$E := \frac{\mathrm{i}^m}{(2\pi)^n} \mathcal{F}\Big(\mathrm{Pf}_{\lambda=0}\Big(\lim_{\epsilon \to 0} (\text{in } \mathcal{S}') |\xi|^\lambda \frac{Y(|P(\xi)| - \epsilon)}{P(\xi)}\Big)\Big).$$

Then the following holds:

(i) $P(\partial)E = \delta$ and $E(-x) = (-1)^m E$;
(ii) $\operatorname{sing\,supp}_{(\mathrm{A})} E = \{t \nabla P(\xi) : \xi \in \mathbf{R}^n, P(\xi) = 0, t \in \mathbf{R}\} =: W$.
(iii) If $|\nu| > m - n$ and $x \in \mathbf{R}^n \setminus W$, then $\partial^\nu E(x) = \frac{1}{2}\langle \widetilde{\alpha_x}, \widetilde{\omega_{x,\nu}}\rangle$,

where $\omega_{x,\nu}$ is as in part II and $\alpha_x = \frac{1}{2}(\beta_x + \overline{\beta_x})$,

$$\beta_x : S_x \longrightarrow \mathbf{CP}^{n-1} : \xi \longmapsto [\xi - \mathrm{i}\, v(\xi)], \qquad S_x := \{\xi \in \mathbf{R}^n : |\xi| = 1, \xi \perp x\},$$

with

$$v(\xi) = \epsilon\left(\nabla P(\xi) - \frac{x}{|x|^2}\langle x, \nabla P(\xi)\rangle\right)$$

and $\epsilon > 0$ small enough.
(iv) If $x \in \mathbf{R}^n \setminus W$ and $\widetilde{\alpha_x} = 0$ (the Petrovsky condition), then E is a polynomial of the degree $m-n$ in the connectivity component of x in $\mathbf{R}^n \setminus W$.

EXAMPLE. Let us consider $n = m = 3$ and $P(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3$ (cf. II, Ex. 2) for $a > -1$. If x is inside the cone L which is the connectivity component of $\mathbf{R}^3 \setminus W$ that contains $(1,1,1)$ (and which is depicted in Fig. 1 in the case of $a = 0$), then β_x, which equals $\overline{\beta_x}$, has the same shape as α_x in Fig. 5, left part. Hence E is constant in the two components of $\mathbf{R}^3 \setminus W$ containing $\pm(1,1,1)$, respectively.

On the calculation of a fundamental solution of $\partial_1^3 + \partial_2^3 + \partial_3^3 + 3a\partial_1\partial_2\partial_3$.

a) The polynomial $P(\xi) := \xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3$ is of principal type for $a \in \mathbf{R} \setminus \{0\}$. For definiteness, we assume $a > -1$. Let the fundamental solution E of $P(\partial)$ be defined as in the proposition and let $L \cup -L$ be the two lacunae of E (cf. the example above). For $x \in \mathbf{R}^3 \setminus (\overline{L} \cup -\overline{L})$, β_x has the same shape as the cycle in the right part of Fig. 5. If $p_x, \overline{p_x}$ are the two non-real zeros of P in $\{[\zeta] \in \mathbf{CP}^2 : \langle \zeta, x\rangle = 0\}$, and if p_x is chosen properly, then part (iii) in the proposition yields

$$\begin{aligned}
\partial_3 E(x) &= -\frac{1}{4\pi^2}\cdot\frac{1}{2}\int_{\beta_x}\left[\frac{\zeta_3\eta_x(\zeta)}{P(\zeta)}\right] \\
&= -\frac{1}{8\pi^2}\cdot 2\pi\mathrm{i}\left(\mathrm{Res}_{\zeta=p_x}\left[\frac{\zeta_3\eta_x(\zeta)}{P(\zeta)}\right] - \mathrm{Res}_{\zeta=\overline{p_x}}\left[\frac{\zeta_3\eta_x(\zeta)}{P(\zeta)}\right]\right) \\
&= \frac{1}{2\pi}\,\mathrm{Im}\left(\mathrm{Res}_{\zeta=p_x}\left[\frac{\zeta_3\eta_x(\zeta)}{P(\zeta)}\right]\right).
\end{aligned}$$

The brackets in these formulae symbolize the passage from one-forms on $\mathbf{C}^3$ to such on $\mathbf{CP}^2$ (cf. [1, II, p. 162]).
b) It is easy to check that $\lambda_x(\zeta) = |x|^{-2}\det(\zeta, x, \mathrm{d}\zeta)$ verifies the condition

$$\mathrm{d}(\langle\zeta, x\rangle) \wedge \lambda_x(\zeta) = \zeta_1 \mathrm{d}\zeta_2 \wedge \mathrm{d}\zeta_3 - \zeta_2 \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_3 + \zeta_3 \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2 + O(\langle\zeta, x\rangle)$$

(cf. II, (iv), (b)). Let us parametrize U_x by $z = \zeta_1/\zeta_3$. Then

$$\left[\frac{\zeta_3 \eta_x(\zeta)}{P(\zeta)}\right] = \left(\frac{\lambda_x(\zeta)}{P(\zeta)}\right)\bigg|_{\zeta_1=z,\,\zeta_2=-(x_1z+x_3)/x_2,\,\zeta_3=1}$$
$$= \frac{1}{|x|^2 R_x(z)} \cdot \det\begin{pmatrix} z & x_1 & \mathrm{d}z \\ -(x_1z+x_3)/x_2 & x_2 & -x_1(\mathrm{d}z)/x_2 \\ 1 & x_3 & 0 \end{pmatrix} = -\frac{\mathrm{d}z}{x_2 R_x(z)},$$

where $R_x(z) := P(z, -(x_1z+x_3)/x_2, 1)$. If $p_x = [z_1(x), z_2(x), 1]$, then we conclude that

$$\mathrm{Res}_{\zeta=p_x}\left[\frac{\zeta_3\eta_x(\zeta)}{P(\zeta)}\right] = -\frac{1}{x_2 R_x'\big(z_1(x)\big)}.$$

c) For x_1, x_2 fixed and $x = (x_1, x_2, x_3)$, let us stress the dependence of R, z_j on the third coordinate by writing $R_{x_3}, z_j(x_3)$, instead of $R_x, z_j(x)$, $j = 1, 2$, respectively. By integration, the above yields

$$E(x) = \text{constant} + \int_0^{x_3} \partial_3 E(x_1, x_2, t)\,\mathrm{d}t = \text{constant } -\frac{1}{2\pi x_2}\,\mathrm{Im}\int_0^{x_3}\frac{\mathrm{d}t}{R_t'\big(z_1(t)\big)}.$$

Let us substitute $z_1 = z_1(t)$ as new variable in the above integral. The identity

$$0 = R_t\big(z_1(t)\big) = P\big(z_1(t), -(x_1z_1(t)+t)/x_2, 1\big)$$

implies

$$0 = \frac{\mathrm{d}R_t\big(z_1(t)\big)}{\mathrm{d}t} = R_t'\big(z_1(t)\big)\cdot\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} - \frac{1}{x_2}\,\partial_2 P\big(z_1(t), -(x_1z_1(t)+t)/x_2, 1\big)$$

and hence

$$\frac{\mathrm{d}t}{R_t'\big(z_1(t)\big)} = -\frac{x_2\,\mathrm{d}z_1}{\partial P/\partial z_2}$$

and

$$E(x) = \text{constant } \pm\frac{1}{2\pi}\,\mathrm{Im}\int_{[\zeta_0]}^{[\zeta(x)]}\Omega,$$

where $\Omega = \dfrac{\mathrm{d}z_1}{\partial P/\partial z_2}$ spans the (one-dimensional) space $\Omega^1(X^c)$ of holomorphic one-forms on the elliptic curve $X^c = \{[\zeta] \in \mathbf{CP}^2 : P(\zeta) = 0\}$, $z_1 = \zeta_1/\zeta_3$, $z_2 = \zeta_2/\zeta_3$, and the integral runs over some path in X^c ending in $[\zeta(x)]$, which satisfies $\langle\zeta(x), x\rangle = 0$. Eventually the addition theorem for elliptic integrals is applied in order to evaluate the imaginary part of the integral. For the final result, we refer to the theorem in [16, p. 286].

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