A Brief History of Fundamental Solutions

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1. Fundamental Solutions in the 18th and 19th Century: Special Equations of Mathematical Physics

The first use of a fundamental solution (in the sequel abbreviated as FS) can probably be ascribed to *Jean d'Alembert*. In 1747 he considered the deflection u of a vibrating string. It satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f,$$

which is solved by convolving f with the FS $E(t,x) = \frac{1}{2c} Y(t - |x|/c)$ of the operator $\partial_t^2 - c^2 \partial_x^2$. In fact, this yields the formula

$$u(t,x) = f * E = \frac{1}{2c} \iint_{|x-\xi|<c(t-\tau)} f(\tau,\xi)\, \mathrm{d}\tau\, \mathrm{d}\xi,$$

which—applied to the initial value problem, i.e. with $f = \delta(t) \otimes u_1(x) + \delta'(t) \otimes u_0(x)$, where $u_j(x) = (\partial_t^j u)(0,x)$, $j = 0,1,$—furnishes d'Alembert's solution, see [A], [Lü, p. 15 ff]. We observe that the FS E appears, as in much of the old literature, only in an indirect way.

Here and in the following, we use the notations $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$, $\partial_1 = \frac{\partial}{\partial x_1}$ etc., $\Delta_n = \partial_1^2 + \cdots + \partial_n^2$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $P(\partial) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \partial^\alpha$, $Y(t) = 1$ for $t \geq 0$ and $Y(t) = 0$ else, $x = (x_1, \ldots, x_n)^T$, $\nabla = (\partial_1, \ldots, \partial_n)^T$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

In 1789, *Pierre Simon de Laplace* used the FS $E = -\frac{1}{4\pi|x|}$ of the elliptic operator Δ_3, which bears his name, and thereby established the connexion of the Laplace-operator with the Newtonian gravitational potential (cf. [L1]). To tell the truth, *Laplace* just recognized that $\Delta_3(E * f) = 0$ outside the support of f, and it was *Simon Denis Poisson*, who obtained the equation $\Delta_3(E * f) = f$ in 1813 (cf. [P1]).

In 1809, *Laplace* considered the first parabolic operator, namely the heat operator $\partial_t - \Delta_n$, and calculated its FS

$$E(t,x) = \frac{Y(t)}{(4\pi t)^{n/2}}\, \mathrm{e}^{-|x|^2/(4t)}$$

in the case of $n = 1$, cf. [L2]. The generalization to higher n, in particular to $n = 2$, was found by *Poisson* in 1818 ([P2]).

In 1818, *Joseph Fourier* was able to calculate the FS E of the operator of the dynamic deflections of beams $\partial_t^2 + \partial_x^4$, an operator of fourth order:

$$E(t,x) = \frac{Y(t)}{2\sqrt{\pi}} \int_0^t \sin\Big(\frac{x^2}{4\tau} + \frac{\pi}{4}\Big) \frac{\mathrm{d}\tau}{\sqrt{\tau}} = Y(t)\Big[\sqrt{\frac{t}{\pi}}\, \sin\Big(\frac{x^2}{4t} + \frac{\pi}{4}\Big) - \frac{|x|}{2}\, C\Big(\frac{x^2}{4t}\Big) + \frac{|x|}{2}\, S\Big(\frac{x^2}{4t}\Big)\Big],$$

$$\text{where } \left.\begin{matrix} C(x) \\ S(x) \end{matrix}\right\} = \frac{1}{\sqrt{2\pi}} \int_0^x \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\}(u) \frac{\mathrm{d}u}{\sqrt{u}},$$

see [F].

As well in 1818, *Poisson* generalized d'Alembert's formula to three space dimensions by representing the solutions of the wave operator $\partial_t^2 - \Delta_3$ as convolution with the FS $E = \delta(t - |x|)/(4\pi|x|)$, cf. [P3]. This notation, viz. the first use of Dirac's delta function, goes back to *Gustav Kirchhoff*'s paper of 1882 (see [K], [Lü, p. 99]).

In 1849, *George Stokes* obtained—as the kernel of an integral representation—the fundamental matrix E of the system of partial differential operators which describes elastic waves in isotropic media ([St]). This system can be found already in a memoir of 1829 by *Poisson* (cf. [P4]). It is given by

$$P(\partial) = (\partial_t^2 - \mu\Delta_3)I_3 - (\lambda + \mu)\nabla \cdot \nabla^T \qquad (I_3 = 3 \text{ by } 3 \text{ unit matrix})$$

and Stokes' fundamental matrix reads

$$E(t,x) = \frac{I_3|x|^2 - x \cdot x^T}{4\pi\mu|x|^3}\, \delta\Big(t - \frac{|x|}{\sqrt{\mu}}\Big) + \frac{x \cdot x^T}{4\pi(\lambda + 2\mu)|x|^3}\, \delta\Big(t - \frac{|x|}{\sqrt{\lambda + 2\mu}}\Big) + $$

$$+ \frac{t}{4\pi|x|^3}\Big(I_3 - \frac{3x \cdot x^T}{|x|^2}\Big)\Big[Y\Big(t - \frac{|x|}{\sqrt{\mu}}\Big) - Y\Big(t - \frac{|x|}{\sqrt{\lambda + 2\mu}}\Big)\Big].$$

The FS $E = Y(t - |x|)/(2\pi\sqrt{t^2 - |x|^2})$ of the wave operator in two space dimensions, i.e. of $\partial_t^2 - \Delta_2$, was found as late as 1894 by *Vito Volterra*, cf. [V].

2. Fundamental Solutions in the 20th Century: General Theories

Investigating the equations of static anisotropic elasticity, *Ivar Fredholm* found in 1900 (cf. [Fr1]) the fundamental matrix E of the elliptic 3 by 3 system

$$P(\partial) = \Big(\sum_{k,l=1}^{3} c_{ijkl}\partial_k\partial_l\Big)_{i,j=1,2,3}, \qquad c_{ijkl} \in \mathbf{R},$$

of linear partial differential operators in three variables with constant coefficients and homogeneous of second order. In different notation, his result is the following (cf. [Fr1, (10), p. 7], [OW3, 3.2.2, (F)]):

$$E(x) = -\frac{\mathrm{i}\,\mathrm{sign}(x_2)}{2\pi} \sum_{k=1}^{3} \frac{|\zeta_k(x)|^2\, P(\zeta_k(x))^{\mathrm{ad}}}{x_2 \frac{\partial \det P}{\partial \xi_1}\big(\zeta_k(x)\big) - x_1 \frac{\partial \det P}{\partial \xi_2}\big(\zeta_k(x)\big)},$$

where $P(\zeta)^{\mathrm{ad}}$ denotes the adjoint matrix of $P(\zeta)$ and $\zeta_k(x) \in \mathbf{C}^3 \setminus \{0\}$ are determined up to complex factors by the conditions

$$x \cdot \zeta_k(x) = 0, \quad \det P(\zeta_k(x)) = 0, \quad \mathrm{Im}\Big(\frac{\zeta_k(x)_1}{\zeta_k(x)_3}\Big) > 0, \quad k = 1, 2, 3.$$

In 1908, *Fredholm* succeeded in representing the FSs of elliptic homogeneous operators in 3 variables by Abelian integrals ([Fr2]). To test the theory, he applied it to the operator $\partial_1^4 + \partial_2^4 + \partial_3^4$, and he obtained—up to the constant factor $-\frac{1}{8\pi}$, which, before the precise definition of FSs (see below), was considered irrelevant—the beautiful formula

$$E(x) = -\frac{1}{8\pi} \sum_{j=1}^{3} |x_j| \int_{\zeta/(2x_j^2)}^{\infty} \frac{\mathrm{d}u}{\sqrt{4u^3 - u}} = -\frac{1}{8\pi} \sum_{j=1}^{3} x_j F\Big(\arcsin\Big(\frac{\sqrt{2}\, x_j}{\sqrt{\zeta + x_j^2}}\Big), \frac{1}{\sqrt{2}}\Big).$$

Therein ζ denotes the largest of the three real roots of the cubic

$$\zeta^3 - (x_1^4 + x_2^4 + x_3^4)\zeta - 2x_1^2 x_2^2 x_3^2 = 0,$$

and F denotes the elliptic integral of the first kind, cf. [GR, 3.131.8 and 8.111]. (A generalization to elliptic operators of the form $\sum_{j,k=1}^{3} c_{jk}\partial_j^2\partial_k^2$ can be found in [W4].)

In 1911, his pupil *Nils Zeilon* gave the first definition of a FS in case it is a locally integrable function (cf. [Z1]):

F is a FS of $f\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big)$ if and only if

$$u = \iiint F(x-\lambda, y-\mu, z-\nu)\phi(\lambda,\mu,\nu)\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu \text{ solves } f\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big)u = \phi.$$

In 1913, *Zeilon* transferred Fredholm's results to non-elliptic operators and considered in particular $\partial_1^3 + \partial_2^3 + \partial_3^3$ (cf. [Z2]). He found the singular support of the FS, but he could not obtain an explicit representation for the FS as Fredholm did in his case. This was done only recently (see [W1], [W2], [W3]).

In three famous papers from 1926 to 1928 (see [He]), *Gustav Herglotz* overcame the restriction to 2 or 3 independent variables and represented the FSs of elliptic and of strictly hyperbolic homogeneous operators of the degree m in n variables (with

$n \leq m$) by $(n-1)-$ fold and by $(n-2)-$ fold integrals, respectively. These formulae later came to be known as the Herglotz-Petrovsky formulae.

In 1945, *Ivan Petrovsky* represented—in the hyperbolic case—the FS E by integrals over cycles in complex projective space and investigated the lacunae of E by means of algebraic topology ([Pe]).

In 1950/51, *Laurent Schwartz* first published his Theory of Distributions ([Sch]), in which framework he also gave the general definition of FSs:

$$E \text{ is a FS of } P(\partial) \text{ if and only if } P(\partial)E = \delta.$$

In Ch. 6 (Transformation de Fourier) of his book, *Schwartz* rederives the FSs of $(\Delta_n - \lambda)^m$, $(\partial_t^2 - \Delta_n - \lambda)^m$, $(\partial_t - \Delta_n - \lambda)^m$, $\lambda \in \mathbf{C}$, by distributional calculus.

In 1952, *Jean Leray* stated a distributional version of the Herglotz-Petrovsky formulae for homogeneous hyperbolic operators, thereby also treating the case $m < n$ (cf. [Le]). The same goal was reached in 1959 by *Vladimir A. Borovikov* for operators of principal type (cf. [B]) and presented in the textbook "Generalized Functions" by *Israel M. Gel'fand and Georgi E. Shilov* (cf. [GS]).

The first existence proofs for FSs were given in 1953/54 by *Bernard Malgrange* and *Leon Ehrenpreis* (cf. [M], [E]). These proofs were based on the Hahn-Banach theorem. In 1957/58, *Lars Hörmander* and *Stanislaw Lojasiewicz* solved independently the "division problem" and proved thereby the existence of temperate FSs (cf. [H1], [Ł]).

In 1970/73, *Michael Atiyah, Raoul Bott, and Lars Gårding* extended and generalized Petrovsky's work, thereby giving a general theory of FSs of hyperbolic operators. For general operators, this was done in the fundamental work of *Lars Hörmander*, cf. [H2], [H3], [H4].

We finally mention the first major table of FSs in 1980 (cf. [O]), the discovery of the connexion of lacunae of FSs with the existence of right inverses by *Reinhold Meise, B. Alan Taylor, and Dietmar Vogt* in 1990 (cf. [MTV]), and *Heinz König*'s proof of the Malgrange-Ehrenpreis theorem by an explicit formula in 1994 (cf. [Kö], [OW1], [OW2]).

Bibliography

[A] d'Alembert, J., *Recherches sur la courbe que forme une corde tendue mise en vibration*, Hist. et Mem. Acad. Sci. Berlin **3** (1747), 214–249.

[ABG] Atiyah, M.F., R. Bott, and L. Gårding, *Lacunas for hyperbolic differential operators with constant coefficients I, II*, Acta Math. **124** (1970), 110–189; **131** (1973), 145–206.

[B] Боровиков, В. А., Фундаментальные решения линейных уравнений в частных производных с постоянными коэфициентами, Труды Моск. матем. об-ва **8** (1959), 199–257; Engl. transl.: V. A. Borovikov, *Fundamental solutions of linear partial differential equations with constant coefficients,* Amer. Math. Soc. Transl., Ser. 2, **25** (1963), 11–66.

[E] Ehrenpreis, L., *Solution of some problems of division. Part I. Division by a polynomial of derivation*, Am. J. Math. **76** (1954), 883–903.

[F] Fourier, J., *Note relative aux vibrations des surfaces élastiques et au mouvement des ondes*, Bull. Sci. Soc. Philomatique Paris (1818), 129–136.

[Fr1] Fredholm, I., *Sur les équations de l'équilibre d'un corps solide élastique*, Acta Math. **23** (1900), 1–42; Œuvres complètes: 17–58.

[Fr2] Fredholm, I., *Sur l'intégrale fondamentale d'une équation différentielle elliptique à coefficients constants*, Rend. Circ. Mat. Palermo **25** (1908), 346–351; Œuvres complètes: 117–122.

[GS] И. М. Гельфанд, и Г. Е. Шилов, Обобщённые функции и действия над ними, Физматгиз, Москва, 1958; Engl. transl.: I. M. Gel'fand and G. E. Shilov, *Generalized Functions I, Properties and Operations*, Academic Press, New York, 1964.

[GR] Gradshteyn, I. S., and I. M. Ryzhik, *Tables of series, products, and integrals*, Harri Deutsch, Thun, 1981.

[He] Herglotz, G., *Über die Integration linearer partieller Differentialgleichungen mit konstanten Koeffizienten I–III*, Ber. Sächs. Akademie Wiss. **78** (1926), 93–126, 287–318; **80** (1928), 69–114.

[H1] Hörmander, L., *On the theory of general partial differential operators*, Acta Math. **94** (1955), 161–248.

[H2] Hörmander, L., *Local and global properties of fundamental solutions*, Math. Scand. **5** (1957), 27–39.

[H3] Hörmander, L., *Linear partial differential operators*, Grundlehren, 116, Springer, Berlin, 1963.

[H4] Hörmander, L., *The analysis of linear partial differential operators. Vol. I–IV*, Grundlehren, 256/7, 274/5, Springer, Berlin, 1983, 1985.

[K] Kirchhoff, G., *Zur Theorie der Lichtstrahlen*, Sitz.ber. K. Preuss. Akad. Wiss. Berlin (1882), 641–669.

[Kö] König, H., *An explicit formula for fundamental solutions of linear partial differential equations with constant coefficients*, Proc. AMS **120** (1994), 1315–1318.

[L1] Laplace, P. S., *Théorie de Jupiter et Saturne*, Mém. Acad. Roy. Sci. Paris (1789), 201–234; cf. also Traité de mécanique céleste II, no. 11.

[L2] Laplace, P. S., J. École Polytéch. cah. 15 (1809), p. 240 (quoted in Enc. Math. Wiss. Band II, 1. Teil, 2. Hälfte, p. 1198).

[Le] Leray, J., *Hyperbolic differential equations*, Institute of Advanced Study, Princeton, 1952.

[Ł] Łojasiewicz, S., *Sur le problème de la division*, Studia Math. **18** (1959), 87–136.

[Lü] Lützen, J., *The prehistory of the theory of distributions*, Springer, Berlin, 1981.

[M] Malgrange, B., *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier **6** (1955/56), 271–355.

[MTV] Meise, R., B. A. Taylor, and D. Vogt, *Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse*, Ann. Inst. Fourier **40** (1990), 619–655.

[O] Ortner, N., *Regularisierte Faltung von Distributionen. Teil 1: Zur Berechnung von Fundamentallösungen. Teil 2: Eine Tabelle von Fundamentallösungen*, ZAMP **31** (1980), 133–173.

[OW1] Ortner, N., and P. Wagner, *A short proof of the Malgrange-Ehrenpreis theorem*, Functional Analysis, Proc. of the 1st Int. Workshop in Trier, ed. by S. Dierolf, S. Dineen, P. Domański, de Gruyter, Berlin, 1996, pp. 343–352.

[OW2] Ortner, N., and P. Wagner, *A survey on explicit representation formulae for fundamental* *solutions of linear partial differential operators*, Acta Applicandae Math. **47** (1997), 101–124.

[OW3] Ortner, N., and P. Wagner, *Fundamental matrices of homogeneous hyperbolic systems. Applications to crystal optics, elastodynamics and piezoelectromagnetism*, manuscript, 2001.

[Pe] Petrovsky, I. G., *On the diffusion of waves and the lacunas for hyperbolic equations*, Mat. Sb. **17** (1945), 283–370.

[P1] Poisson, S. D., *Mémoire sur les intégrales définies*, J. École Polytéch. cah. 9 (1813), 215–246.

[P2] Poisson, S. D., *Sur l'intégrale de l'équation relative aux vibrations des surfaces élastiques et au mouvement des ondes*, Bull. Sci. Soc. Philomatique Paris (1818), 125–128.

[P3] Poisson, S. D., Mém. Acad. Sci. Paris **3** (1818), p. 131 (quoted in Enc. Math. Wiss. Band II, 1. Teil, 2. Hälfte, p. 1258).

[P4] Poisson, S. D., *Mémoire sur l'équilibre et le mouvement des corps élastiques*, Mém. Acad. Sci. Paris **8** (1829), 357–570.

[Sch] Schwartz, L., *Théorie des distributions*, Nouv. éd., Hermann, Paris, 1966.

[St] Stokes, G. G., *On the dynamical theory of diffraction*, Trans. Cambridge Phil. Soc. **9** (1849), 1–62.

[V] Volterra, V., *Sur les vibrations des corps élastiques isotropes*, Acta Math. **18** (1894), 161–232.

[W1] Wagner, P., *Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions*, Acta Math. **182** (1999), 283–300.

[W2] Wagner, P., *A fundamental solution of N. Zeilon's operator*, Math. Scand. **86** (2000), 273–287.

[W3] Wagner, P., *Zeilon's operator and lacunae*, Proc. PDE-Workshop 2001 in Graz, ed. by H. Florian, N. Ortner, F. Schnitzer, W. Tutschke, World Scientific, Singapur, to appear.

[W4] Wagner, P., *On the fundamental solutions of some quartic operators in three dimensions I*, manuscript, 2001.

[Z1] Zeilon, N., *Das Fundamentalintegral der Allgemeinen Partiellen Linearen Differentialgleichung mit konstanten Koeffizienten*, Arkiv f. Mat. Astr. o. Fys. **6** (1911), 1–32.

[Z2] Zeilon, N., *Sur les intégrales fondamentales des équations à charactéristique réelle de la Physique Mathématique*, Arkiv f. Mat. Astr. o. Fys. **9** (1913/14), no. 18, 1–70.