Explicit Calculation of Fundamental Solutions

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I) A BRIEF HISTORY OF FUNDAMENTAL SOLUTIONS

1. Fundamental Solutions in the 18th and 19th Century: Special Equations of Mathematical Physics

The first use of a non-trivial fundamental solution (in the sequel abbreviated as FS) can probably be ascribed to *Jean d'Alembert*. In 1747 he considered the deflection u of a vibrating string. It satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f,$$

which is solved by convolving f with the FS $E(t,x) = \frac{1}{2c} Y(t - |x|/c)$ of the operator $\partial_t^2 - c^2 \partial_x^2$. In fact, this yields the formula

$$u(t,x) = f * E = \frac{1}{2c} \iint_{|x-\xi|<c(t-\tau)} f(\tau,\xi)\, \mathrm{d}\tau\, \mathrm{d}\xi,$$

which—applied to the initial value problem, i.e. with $f = \delta(t) \otimes u_1(x) + \delta'(t) \otimes u_0(x)$, where $u_j(x) = (\partial_t^j u)(0,x)$, $j = 0,1,$—furnishes d'Alembert's solution, see [A], [Lü, p. 15 ff]. We observe that the FS E appears, as in much of the old literature, only in an implicit way.

Here and in the following, we use the notations $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$, $\partial_1 = \frac{\partial}{\partial x_1}$ etc., $\Delta_n = \partial_1^2 + \cdots + \partial_n^2$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $P(\partial) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \partial^\alpha$, $Y(t) = 1$ for $t \geq 0$ and $Y(t) = 0$ else, $x = (x_1, \ldots, x_n)^T$, $\nabla = (\partial_1, \ldots, \partial_n)^T$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

In 1789, *Pierre Simon de Laplace* used the FS $E = -\frac{1}{4\pi|x|}$ of the elliptic operator Δ_3, which bears his name, and thereby established the connexion of the Laplace operator with the Newtonian gravitational potential (cf. [L1]). To tell the truth, *Laplace* just recognized that $\Delta_3(E * f) = 0$ outside the support of f, and it was *Simon Denis Poisson*, who obtained the equation $\Delta_3(E * f) = f$ in 1813 (cf. [P1]).

In 1809, *Laplace* considered the first parabolic operator, namely the heat operator $\partial_t - \Delta_n$, and calculated its FS

$$E(t,x) = \frac{Y(t)}{(4\pi t)^{n/2}}\, \mathrm{e}^{-|x|^2/(4t)}$$

in the case $n = 1$, cf. [L2]. The generalization to higher n, in particular to $n = 2$, was found by *Poisson* in 1818 ([P2]).

In 1818, *Joseph Fourier* was able to calculate the FS E of the operator of the dynamic deflections of beams $\partial_t^2 + \partial_x^4$, an operator of fourth order:

$$E(t,x) = \frac{Y(t)}{2\sqrt{\pi}} \int_0^t \sin\Big(\frac{x^2}{4\tau} + \frac{\pi}{4}\Big)\frac{d\tau}{\sqrt{\tau}}$$
$$= Y(t)\Big[\sqrt{\frac{t}{\pi}}\,\sin\Big(\frac{x^2}{4t}+\frac{\pi}{4}\Big) - \frac{|x|}{2}\,C\Big(\frac{x^2}{4t}\Big) + \frac{|x|}{2}\,S\Big(\frac{x^2}{4t}\Big)\Big],$$
$$\text{where}\quad \left.\begin{matrix} C(x)\\ S(x)\end{matrix}\right\} = \frac{1}{\sqrt{2\pi}}\int_0^x \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(u)\frac{du}{\sqrt{u}},$$

see [F].

As well in 1818, *Poisson* generalized d'Alembert's formula to three space dimensions by representing the solutions of the wave operator $\partial_t^2 - \Delta_3$ as convolution with the FS $E = \delta(t-|x|)/(4\pi|x|)$, cf. [P3]. This notation, viz. the first use of Dirac's delta function, goes back to *Gustav Kirchhoff*'s paper of 1882 (see [K], [Lü, p. 99]).

In 1849, *George Stokes* obtained—as the kernel of an integral representation—the fundamental matrix E of the system of partial differential operators which describes elastic waves in isotropic media ([St]). This system can be found already in a memoir of 1829 by *Poisson* (cf. [P4]). It is given by

$$P(\partial) = (\partial_t^2 - \mu\Delta_3)I_3 - (\lambda+\mu)\nabla\cdot\nabla^T \qquad (I_3 = 3 \text{ by } 3 \text{ unit matrix})$$

and Stokes' fundamental matrix reads

$$E(t,x) = \frac{I_3|x|^2 - x\cdot x^T}{4\pi\mu|x|^3}\,\delta\Big(t-\frac{|x|}{\sqrt{\mu}}\Big) + \frac{x\cdot x^T}{4\pi(\lambda+2\mu)|x|^3}\,\delta\Big(t-\frac{|x|}{\sqrt{\lambda+2\mu}}\Big) +$$
$$+ \frac{t}{4\pi|x|^3}\Big(I_3 - \frac{3x\cdot x^T}{|x|^2}\Big)\Big[Y\Big(t-\frac{|x|}{\sqrt{\mu}}\Big) - Y\Big(t-\frac{|x|}{\sqrt{\lambda+2\mu}}\Big)\Big].$$

The FS $E = Y(t-|x|)/(2\pi\sqrt{t^2-|x|^2})$ of the wave operator in two space dimensions, i.e. of $\partial_t^2 - \Delta_2$, was found as late as 1894 by *Vito Volterra*, cf. [V].

2. Fundamental Solutions in the 20th Century: General Theories

Investigating the equations of static anisotropic elasticity, *Ivar Fredholm* found in 1900 (cf. [Fr1]) the fundamental matrix E of the elliptic 3 by 3 system

$$P(\partial) = \Big(\sum_{k,l=1}^{3} c_{ijkl}\partial_k\partial_l\Big)_{i,j=1,2,3}, \qquad c_{ijkl} \in \mathbf{R},$$

of linear partial differential operators in three variables with constant coefficients and homogeneous of second order. In our notation, his result is the following (cf. [Fr1, (10), p. 7], [OW3, 3.2.2, (F)]):

$$E(x) = -\frac{i\,\mathrm{sign}(x_2)}{2\pi}\sum_{k=1}^{3}\frac{|\zeta_k(x)|^2\,P(\zeta_k(x))^{\mathrm{ad}}}{x_2\frac{\partial\det P}{\partial\xi_1}\big(\zeta_k(x)\big) - x_1\frac{\partial\det P}{\partial\xi_2}\big(\zeta_k(x)\big)},$$

where $P(\zeta)^{\mathrm{ad}}$ denotes the adjoint matrix of $P(\zeta)$ and $\zeta_k(x) \in \mathbf{C}^3\backslash\{0\}$ are determined up to complex factors by the conditions

$$x \cdot \zeta_k(x) = 0, \quad \det P(\zeta_k(x)) = 0, \quad \mathrm{Im}\Big(\frac{\zeta_k(x)_1}{\zeta_k(x)_3}\Big) > 0, \quad k = 1, 2, 3.$$

In 1908, *Fredholm* succeeded in representing the FSs of elliptic homogeneous operators in 3 variables by Abelian integrals ([Fr2]). To test the theory, he applied it to the operator $\partial_1^4 + \partial_2^4 + \partial_3^4$, and he obtained, up to the constant factor $-\frac{1}{8\pi}$, the beautiful formula

$$\begin{aligned} E(x) &= -\frac{1}{8\pi}\sum_{j=1}^{3}|x_j|\int_{\zeta/(2x_j^2)}^{\infty}\frac{\mathrm{d}u}{\sqrt{4u^3-u}} \\ &= -\frac{1}{8\pi}\sum_{j=1}^{3}x_j F\Big(\arcsin\Big(\frac{\sqrt{2}\,x_j}{\sqrt{\zeta+x_j^2}}\Big), \frac{1}{\sqrt{2}}\Big). \end{aligned}$$

Therein ζ denotes the largest of the three real roots of the cubic

$$\zeta^3 - (x_1^4 + x_2^4 + x_3^4)\zeta - 2x_1^2x_2^2x_3^2 = 0,$$

and F denotes the elliptic integral of the first kind, cf. [GR, 3.131.8 and 8.111]. (A generalization to elliptic operators of the form $\sum_{j,k=1}^{3} c_{jk}\partial_j^2\partial_k^2$ can be found in [W4].)

In 1911, his pupil *Nils Zeilon* gave the first definition of a FS in case it is a locally integrable function (cf. [Z1]):

$$F \text{ is a FS of } f\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big) \text{ if and only if}$$

$$u = \iiint F(x-\lambda, y-\mu, z-\nu)\phi(\lambda,\mu,\nu)\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu$$

$$\text{solves } f\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big)u = \phi.$$

In 1913, *Zeilon* transferred Fredholm's results to non-elliptic operators and, in particular, he considered $\partial_1^3 + \partial_2^3 + \partial_3^3$ (cf. [Z2]). He determined the singular support of its FS, but, in contrast to Fredholm, he was not able to obtain an explicit representation for the FS. However, explicit formulae were found recently, see [W1], [W2], [W3].

In three famous papers from 1926 to 1928 (see [He]), *Gustav Herglotz* overcame the restriction to 2 or 3 independent variables and represented the FSs of elliptic and of strictly hyperbolic homogeneous operators of the degree m in n variables (with $n \le m$) by $(n-1)-$ fold and by $(n-2)-$ fold integrals, respectively. Later, these formulae came to be known as the Herglotz-Petrovsky formulae.

In 1945, *Ivan Petrovsky* represented—in the hyperbolic case—the FS E by integrals over cycles in complex projective space and investigated the lacunas of E by means of algebraic topology ([Pe]).

In 1950/51, *Laurent Schwartz* first published his Theory of Distributions ([Sch]), in which framework he also gave the general definition of FSs:

$$E \text{ is a FS of } P(\partial) \text{ if and only if } P(\partial)E = \delta.$$

In Ch. 6 (Transformation de Fourier) of his book, *Schwartz* rederives the FSs of $(\Delta_n - \lambda)^m$, $(\partial_t^2 - \Delta_n - \lambda)^m$, $(\partial_t - \Delta_n - \lambda)^m$, $\lambda \in \mathbf{C}$, by distributional calculus.

In 1952, *Jean Leray* stated a distributional version of the Herglotz-Petrovsky formulae for homogeneous hyperbolic operators, thereby also treating the case $m < n$ (cf. [Le]). The same goal was reached in 1959 by *Vladimir A. Borovikov* for operators of principal type (cf. [B]) and presented in the textbook "Generalized Functions" by *Israel M. Gel'fand and Georgi E. Shilov* (cf. [GS]).

The first existence proofs for FSs were given in 1953/54 by *Bernard Malgrange* and *Leon Ehrenpreis* (cf. [M], [E]). These proofs were based on the Hahn-Banach theorem. In 1957/58, *Lars Hörmander* and *Stanislaw Łojasiewicz* independently solved the "division problem" and thereby proved the existence of temperate FSs (cf. [H2], [Ł]).

In 1970/73, *Michael Atiyah, Raoul Bott, and Lars Gårding* extended and generalized Petrovsky's work, thereby developing a general theory of FSs of hyperbolic operators, cf. [ABG]. For general operators, this was established in the fundamental work of *Lars Hörmander*, cf. [H1], [H3], [H4].

We also mention the first major table of FSs in 1980 (cf. [O]) and the discovery of the connexion of lacunas of FSs with the existence of right inverses by *Reinhold Meise, B. Alan Taylor, and Dietmar Vogt* in 1990 (cf. [MTV]).

Finally, I would like to sketch a proof of the Malgrange-Ehrenpreis theorem I found in 1994, influenced by a paper of *Heinz König* (cf. [Kö]). This constructive proof seems to be the shortest one at present.

Theorem. (*Malgrange/Ehrenpreis,* 1953/54)

Let $P(\xi) = \sum_{|\alpha| \le m} c_\alpha \xi^\alpha$ *be a not identically vanishing polynomial in* $\mathbf{R}^n$, (*i.e.* $c_\alpha \in \mathbf{C}$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, *not all* $c_\alpha = 0$). *Then there exists a FS of* $P(\partial)$, *i.e.,* $\exists E \in \mathcal{D}'(\mathbf{R}^n) : P(\partial)E = \delta$.

Proof. ([OW1]) The distribution $E \in \mathcal{D}'(\mathbf{R}^n)$ defined by

$$E(x) = \frac{1}{\overline{P_m(\eta)}} \int_{\lambda \in \mathbb{C}, |\lambda|=1} \lambda^m \mathrm{e}^{\lambda \eta x} \mathcal{F}^{-1} \left(\frac{\overline{P(\mathrm{i}\xi + \lambda\eta)}}{P(\mathrm{i}\xi + \lambda\eta)} \right) \frac{\mathrm{d}\lambda}{2\pi \mathrm{i} \lambda}$$

is a FS of $P(\partial)$, if $P_m(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha$ (i.e. P_m is the principal part of P), $\eta \in \mathbf{C}^n$ with $P_m(\eta) \ne 0$ is fixed, $\eta x = \eta_1 x_1 + \cdots + \eta_n x_n$, and $\mathcal{F}$ denotes the Fourier transform $((\mathcal{F}\phi)(x) = \int \phi(\xi)\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}\xi$ for $\phi \in \mathcal{D}$, and extended to $\mathcal{S}'$ by continuity) with the inverse $\mathcal{F}^{-1}T = (2\pi)^{-n}(\mathcal{F}T)(-x)$. The formula makes sense, since

$$\frac{\overline{P(\mathrm{i}\xi + \lambda\eta)}}{P(\mathrm{i}\xi + \lambda\eta)} \in L^\infty(\mathbb{R}^n_\xi) \subset \mathcal{S}'(\mathbb{R}^n),$$

and since this distribution continuously depends on λ. That the formula yields a FS is seen by direct verification:

$$\begin{aligned}
P(\partial)E &= \frac{1}{\overline{P_m(\eta)}} \int_{\lambda\in\mathbb{C},|\lambda|=1} \lambda^m P(\partial)\Big(\mathrm{e}^{\lambda\eta x}\mathcal{F}^{-1}\Big(\frac{\overline{P(\mathrm{i}\xi+\lambda\eta)}}{P(\mathrm{i}\xi+\lambda\eta)}\Big)\Big)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} \\
&= \frac{1}{\overline{P_m(\eta)}} \int_{\lambda\in\mathbb{C},|\lambda|=1} \lambda^m \mathrm{e}^{\lambda\eta x}\Big(P(\partial+\lambda\eta)\,\mathcal{F}^{-1}\Big(\frac{\overline{P(\mathrm{i}\xi+\lambda\eta)}}{P(\mathrm{i}\xi+\lambda\eta)}\Big)\Big)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} \\
&= \frac{1}{\overline{P_m(\eta)}} \int_{\lambda\in\mathbb{C},|\lambda|=1} \lambda^m \mathrm{e}^{\lambda\eta x}\mathcal{F}^{-1}\big(\overline{P(\mathrm{i}\xi+\lambda\eta)}\big)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} \\
&= \frac{1}{\overline{P_m(\eta)}} \int_{\lambda\in\mathbb{C},|\lambda|=1} \lambda^m \mathrm{e}^{\lambda\eta x}\overline{P(-\partial+\lambda\eta)}\,\delta\,\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} \\
&= \frac{1}{\overline{P_m(\eta)}} \int_{\lambda\in\mathbb{C},|\lambda|=1} \lambda^m \mathrm{e}^{\lambda\eta x}\Big[\overline{\lambda^m}\,\overline{P_m(\eta)}\,\delta + \sum_{k=0}^{m-1}\overline{\lambda^k}\,Q_k(\partial)\delta\Big]\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} = \delta. \quad \square
\end{aligned}$$

II) DUALITY AND MICROLOCAL ANALYSIS

An important step in the calculation of FSs consists in the determination of its singular support. Here I would like to sketch a connection of microlocal analysis with Plücker's theory of dual algebraic curves. This relation is also at the heart of the Atiyah-Bott-Gårding theory, but applies to non-hyperbolic operators as well.

Let $P(\partial)$ be a real homogeneous operator of the degree m and of *principal type*, i.e., $\forall\xi\in\mathbf{R}^n\setminus\{0\}:\mathrm{d}P(\xi)\neq 0$. Then we can easily solve the division problem $P(\omega)\cdot\Phi=1$ on the sphere $\mathbf{S}^{n-1}$ by $\Phi=\mathrm{vp}\Big(\frac{1}{P(\omega)}\Big)\in\mathcal{D}'(\mathbf{S}^{n-1})$, which is defined through

$$\langle\psi,\Phi\rangle=\lim_{\epsilon\searrow 0}\int_{|P(\omega)|>\epsilon}\frac{\psi(\omega)}{P(\omega)}\,\mathrm{d}o(\omega),\qquad \psi\in\mathcal{D}(\mathbf{S}^{n-1}).$$

From this we obtain $T\in\mathcal{S}'(\mathbf{R}^n)$ with $P(\xi)\cdot T=1$ by putting $T=\operatorname*{Pf}_{\lambda=-m}\big[\Phi\big(\frac{\xi}{|\xi|}\big)|\xi|^\lambda\big]$. The distribution T is homogeneous in $\mathbf{R}^n\setminus\{0\}$ and we can describe its wave front set $\mathrm{WF}\,T$ quite explicitly: Near a zero $\xi_0\in\mathbf{R}^n\setminus\{0\}$ of P, we use $y_1=P(\xi)$ as a coordinate and obtain, since $\mathrm{WF}\,T$ is defined intrinsically in the cotangent space $T^*\mathbf{R}^n$, that

$$\mathrm{WF}\,T\cap T^*(\mathbf{R}^n\setminus\{0\})=\{(\xi,x);\,\xi\in\mathbf{R}^n\setminus\{0\},P(\xi)=0,x=t\cdot\mathrm{d}P(\xi),t\in\mathbf{R}\setminus\{0\}\}.$$

Making use of the following theorem, which goes back to Sato, we obtain a precise description of $\mathrm{WF}\,E$, where $E:=(\mathrm{i}^m/(2\pi)^n)\mathcal{F}T$ is a FS of $P(\partial)$.

Theorem. (Thm. 8.1.8, [H4]) *Let $u\in\mathcal{D}'(\mathbf{R}^n)$ be homogeneous in $\mathbf{R}^n\setminus\{0\}$ and identify $T^*\mathbf{R}^n$ with $\mathbf{R}^{2n}$. Then*

$$\begin{aligned}
(x,\xi)\in\mathrm{WF}(u) &\Longleftrightarrow (\xi,-x)\in\mathrm{WF}(\mathcal{F}u) && \text{if } \xi\neq 0, x\neq 0,\\
x\in\operatorname{supp}u &\Longleftrightarrow (0,-x)\in\mathrm{WF}(\mathcal{F}u) && \text{if } x\neq 0,\\
\xi\in\operatorname{supp}\mathcal{F}u &\Longleftrightarrow (0,\xi)\in\mathrm{WF}(u) && \text{if } \xi\neq 0.
\end{aligned}$$

Hence we conclude that

$$\mathrm{WF}\,E=\{(x,\xi);\,\xi\in\mathbf{R}^n\setminus\{0\},x=0\text{ or }[P(\xi)=0,x=t\cdot\nabla P(\xi),t\in\mathbf{R}\setminus\{0\}]\},$$

where $\nabla P = (\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_n})^T$. In particular,

$$\operatorname{sing\,supp} E = \{t \cdot \nabla P(\xi);\ t \in \mathbf{R}, \xi \in \mathbf{R}^n, P(\xi) = 0\}.$$

This means that $\operatorname{sing\,supp} E$ is the algebraic variety dual to the zero variety of P. Let us recapitulate this concept from algebraic geometry.

If V is a finite dimensional vector space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, then the corresponding projective space is the set of all one-dimensional subspaces in V, i.e.

$$\mathbf{P}(V) = \{[v];\ v \in V \setminus \{0\}\}, \qquad [v] = \mathbf{K} \cdot v.$$

The projective space $\mathbf{P}(V^*)$ is canonically identified with the set of all subspaces of V of codimension one and is called the *dual* projective space. If $X \subset \mathbf{P}(V)$ is a hypersurface given as the zero-set of a homogeneous polynomial P as above, i.e. $X = \{[v] \in \mathbf{P}(V);\ P(v) = 0\}$, then the set of tangent planes to X is an algebraic variety in $\mathbf{P}(V^*)$, called the *dual hypersurface* X^*. We consider these varieties over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and denote them by X or X^c, X^* or X^{c*}, respectively. From the above discussion, we obtain in our case $X^* = \{[x];\ x \in \operatorname{sing\,supp} E \setminus \{0\}\}$.

Trivial example: The cubic $t = s^3$ (written projectively as $x_1^3 - x_2 x_3^2 = 0$ with $s = x_1/x_3, t = x_2/x_3$) has a flex at $s = t = 0$. The dual curve is by definition the collection of all tangent lines, i.e., $t = 3s_0^2(s - s_0) + s_0^3 = ks + d$ and hence is parametrized by $k = 3s_0^2, d = -2s_0^3$. This is Neill's parabola, which has a cusp at the point $k = d = 0$ corresponding to $s = t = 0$.

In general, flexes and cusps correspond to one another by duality in the case of plane curves. If κ, δ, b, f and $\kappa^*, \delta^*, b^*, f^*$ denote the number of cusps, (ordinary) double points, bitangents, flexes of a plane algebraic curve and of its dual, respectively, then the classical Plücker formulae say (cf. [GH, p. 280])

$$b = \delta^*, b^* = \delta, f = \kappa^*, f^* = \kappa, d^* = d(d-1) - 2\delta - 3\kappa, g = \binom{d-1}{2} - \delta - \kappa.$$

Here d, d^* are the degrees of our curves and g denotes the genus.

If X^c is given by $P(\zeta_1, \zeta_2, \zeta_3) = 0$, we obtain X^{c*} as the set of those projective points $[z] \in \mathbf{P}(\mathbf{C}^3)$ where the two equations $\zeta \cdot z = 0, P(\zeta) = 0$ have a multiple projective solution $[\zeta]$, and thus from the zero set of of the discriminant of $P(u, -(uz_1 + z_3)/z_2, 1)$ with respect to u.

III) HOMOGENEOUS CUBIC AND QUARTIC OPERATORS IN 3D

As mentioned earlier, I. Fredholm calculated the FS of $\partial_1^4 + \partial_2^4 + \partial_3^4$, whereas N. Zeilon failed to find an explicit representation for a FS of $\partial_1^3 + \partial_2^3 + \partial_3^3$. In later years, Herglotz, Petrovsky, Garnir etc. explicitly calculated FSs for products of wave and Laplace operators, but, up to 1997, $\partial_1^4 + \partial_2^4 + \partial_3^4$ remained the only irreducible homogeneous operator of degree > 2, the FS of which was known. In 1997, I succeeded in representing a FS of $\partial_1^3 + \partial_2^3 + \partial_3^3$ (which I called "Zeilon's operator") by elliptic integrals, and in 1998, I generalized the result to operators of the form $\partial_1^3 + \partial_2^3 + \partial_3^3 + 3a\partial_1\partial_2\partial_3$, $a \in \mathbf{R} \setminus \{-1\}$ (cf. [W1], [W2]). Let me describe the main result.

According to Newton's classification of real elliptic curves, the non-singular real homogeneous polynomials $P(\xi)$ of third order in three variables are divided into two types according to whether the real projective curve $\{[\xi]\in \mathbf{P}(\mathbf{R}^3): P(\xi)=0\}$ consists of one or of two connected components, respectively. In Hesse's normal form, all non-singular real cubic curves are—up to linear transformations—given by

$$P_a(\xi)=\xi_1^3+\xi_2^3+\xi_3^3+3a\xi_1\xi_2\xi_3, \qquad a\in\mathbf{R}\setminus\{-1\}.$$

(Intuitively, this comes from the fact that a homogeneous cubic polynomial in 3 variables, i.e., $P(\xi)=\sum_{\alpha\in\mathbf{N}_0^3,|\alpha|=3} c_\alpha\xi^\alpha$ has $\binom{3+2}{3}=10$ coefficients and $\dim \mathrm{gl}(\mathbf{R}^3)=9$ and hence the (Teichmüller) space of elliptic curves is one-dimensional.) Let $X_a := \{[\xi]\in\mathbf{P}(\mathbf{R}^3): P_a(\xi)=0\}$ denote the real projective variety defined by P_a. For $a>-1$, X_a is connected, whereas, for $a<-1$, X_a consists of two components (cf. Fig. 1). The corresponding operators $P_a(\partial)$ also differ from the physical viewpoint: For $a<-1$, every projective line through $[1,1,1]$ intersects X_a in three different projective points and thus P_a is strongly hyperbolic in the direction $(1,1,1)$, for $a>-1$, P_a is not hyperbolic in any direction, nor is it an evolution operator.

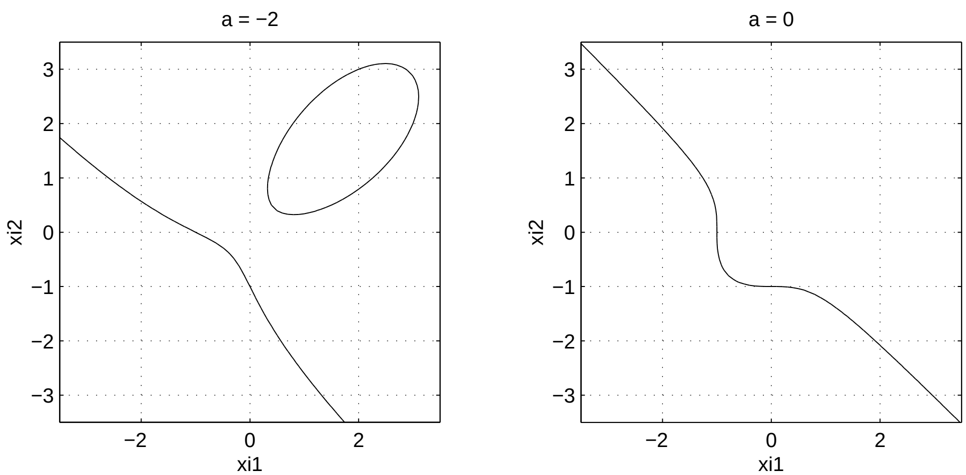

Figure 1: $\{(\xi_1,\xi_2):[\xi_1,\xi_2,1]\in X_a\}$ for $a=-2$ and for $a=0$

We define the fundamental solution E_a of $P_a(\partial)$ as the Fourier transform of the homogeneous distribution which is of order -3 and has $\mathrm{vp}\,\frac{1}{P_a(\omega)}\in\mathcal{D}'(\mathbf{S}^2)$ as its restriction to the sphere. According to II), the (analytic) singular support of E_a is the dual curve of X_a, i.e.,

$$\operatorname{sing\,supp} E_a=\operatorname{sing\,supp}_{\mathrm{A}} E_a=\{t\nabla P_a(\xi);\, t\in\mathbf{R},\, \xi\in\mathbf{R}^3,\, P_a(\xi)=0\}.$$

By the classical Plücker formulae, $[\operatorname{sing\,supp} E_a\setminus\{0\}]$ is an algebraic curve of degree 6. Its complexification has nine cusps, three of which are real in correspondence with the three flexes of X_a (cf. Fig. 2). Explicitly, we have $\operatorname{sing\,supp} E_a=\{x\in\mathbf{R}^3;\, A_a(x)=0\}$, where

$$(1)\quad A_a(x):=3a(a^3+4)x_1^2x_2^2x_3^2+4(a^3+1)(x_1^3x_2^3+x_1^3x_3^3+x_2^3x_3^3)$$
$$+6a^2x_1x_2x_3(x_1^3+x_2^3+x_3^3)-(x_1^3+x_2^3+x_3^3)^2.$$

If $a<-1$, then P_a is hyperbolic with respect to $(1,1,1)$, and

$$(2_1)\qquad W_a:=\{x\in\mathbf{R}^3;\, A_a(x)=0,\, x_1+x_2+x_3\geq 0\}\qquad (a<-1)$$

consists of *two* conical surfaces which are the respective duals of the two components of X_a. Let F_a denote the unique fundamental solution of $P_a(\partial)$ with support in $\{x \in \mathbf{R}^3 : x_1+x_2+x_3 \geq 0\}$. Then $E_a = \frac{1}{2}(F_a - \check{F}_a)$, where the superscript $\check{}$ indicates reflection with respect to the origin. Further, we denote by K_a the propagation cone of P_a with respect to $(1,1,1)$, i.e.,

$$
\begin{aligned}
(3) \qquad K_a &:= \text{dual cone of the component of } (1,1,1) \text{ in } \{x \in \mathbf{R}^3;\ P_a(x) \neq 0\} \\
&= \text{convex hull of } W_a.
\end{aligned}
$$

From the Herglotz-Petrovsky-Leray formula, we infer that F_a has a Petrovsky lacuna inside the cone

$$(4_1) \qquad L_a := \{x \in K_a;\ A_a(x) > 0\} \qquad (a < -1).$$

Hence W_a consists of ∂K_a and of ∂L_a, which bound a convex and a non-convex cone, respectively (cf. Fig. 2).

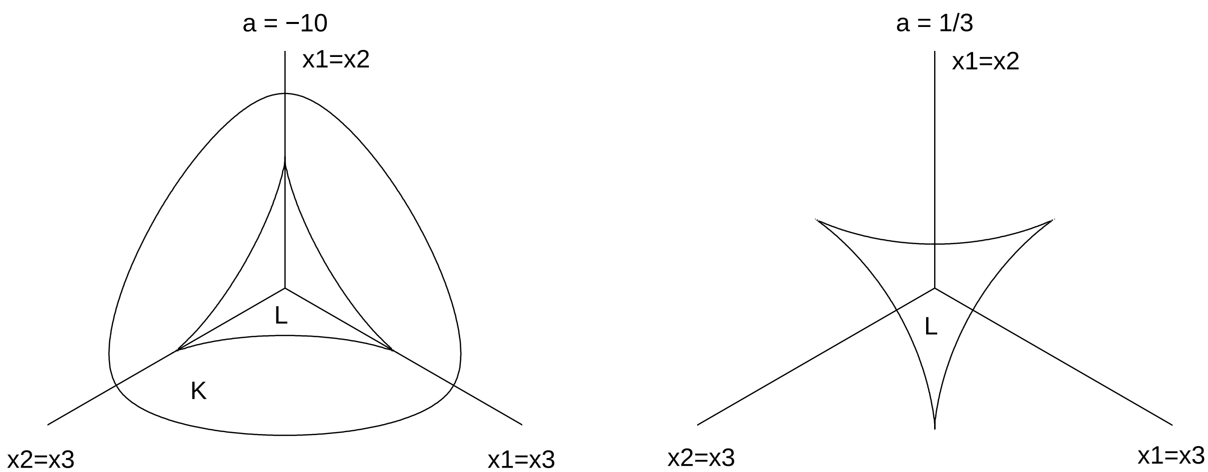

Figure 2: $\{x \in W_a : x_1 + x_2 + x_3 = 1\}$ for $a = -10$ and for $a = \frac{1}{3}$

If $a > -1$, then still E_a has lacunas inside L_a and $-L_a$, where now we define

$$(4_2) \qquad L_a := \text{component of } (1,1,1) \text{ in } \{x \in \mathbf{R}^3;\ A_a(x) > 0\} \qquad (a > -1)$$

and

$$(2_2) \qquad W_a := \partial L_a \qquad (a > -1).$$

In both cases, the fundamental solutions E_a are constant inside L_a and $-L_a$, and we represent these constant values as *complete* elliptic integrals of the first kind. Moreover, E_a is continuous outside the origin.

Outside the lacunas, $E_a(x)$ can be represented by elliptic integrals of the first kind. The final result is contained in the following theorem.

Theorem. *Let* $a \in \mathbf{R} \setminus \{-1\}$. *The limit*

$$T_a := \lim_{\epsilon \searrow 0} \frac{Y(|\xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3| - \epsilon)}{\xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3}$$

defines a distribution in $\mathcal{S}'(\mathbf{R}^3)$*. If* $E_a := \left(\frac{\mathrm{i}}{2\pi}\right)^3 \mathcal{F}T_a$, *and* A_a, W_a, L_a, *and, for* $a < -1$, K_a *are as in* (1), (2$_1$), (2$_2$), (4$_1$), (4$_2$), (3), *respectively, then*

(a) E_a *is a fundamental solution of* $\partial_1^3 + \partial_2^3 + \partial_3^3 + 3a\partial_1\partial_2\partial_3$;
(b) E_a *is homogeneous of degree* 0;
(c) E_a *is odd and invariant under permutations of the co-ordinates*;
(d) $\operatorname{sing\,supp} E_a = \operatorname{sing\,supp}_{\mathrm{A}} E_a = W_a \cup -W_a$;
(e) E_a *is continuous in* $\mathbf{R}^3 \setminus \{0\}$;
(f) *if* $a < -1$, *then* $E_a = \frac{1}{2}(F_a - \check{F}_a)$, $P_a(\partial)F_a = \delta$, $\operatorname{supp} F_a = K_a$;
(g) E_a *is constant in* L_a *and in* $-L_a$, *and the values* $E_a|_{L_a}$ *are given by the following complete elliptic integrals of the first kind:*

$$E_a|_{L_a} = -\frac{1}{4\sqrt{3}\,\pi}\begin{cases} \displaystyle\int_\rho^\infty \frac{\mathrm{d}u}{\sqrt{p_a(u)}} & : a > -1, \\ \displaystyle\int_{-\infty}^\rho \frac{2\,\mathrm{d}u}{\sqrt{p_a(u)}} & : a < -1, \end{cases}$$

where $p_a(u) := 4(a^3+1)u^3 + 9a^2u^2 + 6au + 1$ *and* ρ *is the smallest real root of* $p_a(u)$;

(h) *let* $x \in U_a$, *where* $U_a := \mathbf{R}^3 \setminus (\overline{L}_a \cup -\overline{L}_a)$ *if* $a > -1$ *and* $U_a := K_a \setminus (L_a \cup W_a)$ *if* $a < -1$, *and denote by* $z(x)$ *the only simple real root or, if* x *belongs to one of the co-ordinate axes, the triple root* 0, *respectively, of the cubic equation*

$$\begin{aligned} Q_a(x,z) := A_a(x)z^3 &+ 9(ax_1^2 + x_2x_3)(ax_2^2 + x_1x_3)(ax_3^2 + x_1x_2)z^2 \\ &+ [9a^2x_1^2x_2^2x_3^2 + 6a(x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) + 3x_1x_2x_3(x_1^3 + x_2^3 + x_3^3)]z \\ &+ 3ax_1^2x_2^2x_3^2 + x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3 = 0. \end{aligned}$$

Then z *is a real-analytic function in* U_a, *and*

$$E_a(x) = \frac{Y(-1-a)}{2}\,E_a|_{L_a} + \frac{\operatorname{sign}(\tilde{P}_a(x))}{4\sqrt{3}\,\pi}\int_\rho^{z(x)} \frac{\mathrm{d}u}{\sqrt{p_a(u)}}$$

where $\tilde{P}_a(x) := 3[(a^3-2)\rho + a^2]x_1x_2x_3 - (3a\rho+1)(x_1^3 + x_2^3 + x_3^3)$.

Sketch of the proof. Applying the residue theorem in the Herglotz-Petrovsky-Leray formula and using some substitution yields

(5) $$E_a(x) = C_1 + C_2 \cdot \operatorname{Im}\int_{\gamma(x)} \Omega, \qquad x \in U_a,$$

where C_1, C_2 are constants, $\gamma(x)$ is a path in the elliptic curve $X_a^c := \{[\zeta] \in \mathbf{P}(\mathbf{C}^3); P_a(\zeta) = 0\}$ starting at some fixed point and leading to $[y(x)] \in X_a^c$ defined by $x \cdot y(x) = 0$ and $\operatorname{Im} y_1(x) > 0$, say. Furthermore, Ω is a generator of the space of holomorphic one-forms on X_a^c. Then the addition theorem for elliptic functions (resp. Abel's theorem for elliptic curves) is applied. $\square$

Remarks. Interestingly, it follows from this theorem that the level surfaces of E_a are algebraic. Up to present, there is no theoretical explanation for this fact.

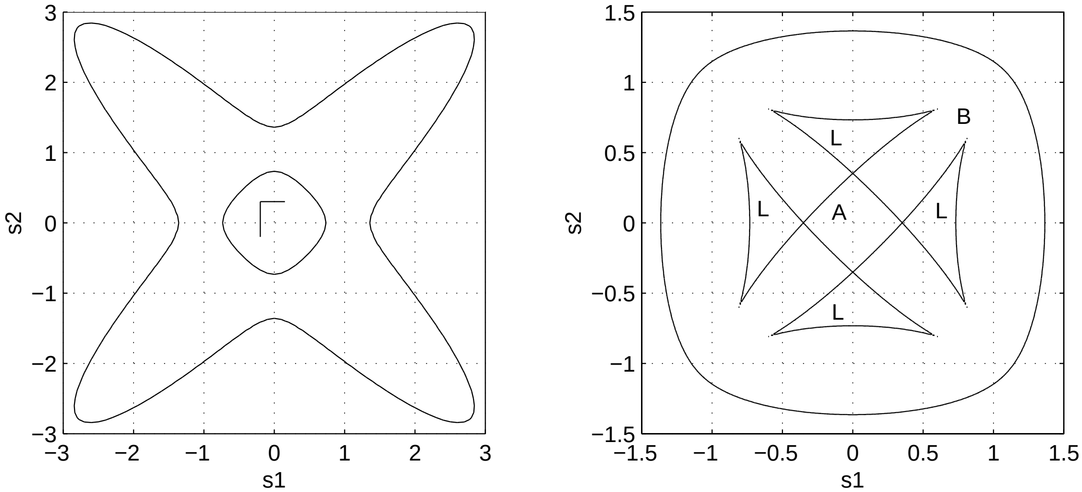

Figure 3: X and W for $\partial_1^4 + \partial_2^4 + \partial_3^4 + 2a\partial_1^2\partial_2^2 + 2b\partial_3^2(\partial_1^2 + \partial_2^2)$ with $a = -0.7$ and $b = -1.2$

In the papers [W4], [W5], we deduce similar formulae for elliptic resp. hyperbolic quartic operators of the form $P(\partial) = \sum_{j,k=1}^{3} c_{jk}\partial_j^2\partial_k^2$. A typical picture of the slowness surface X and of the wave front surface W for such an operator is given in Fig. 3.

In the regions A, B, the fundamental solution E is given by incomplete elliptic integrals of the first kind, in the Petrovsky lacunas L it is given by linear functions the coefficients of which are complete elliptic integrals of the first kind. We refer to [W5] for details.

Note that the Riemann surfaces defined by $P(z) = 0$ in this case have genus 3, but E is still given by *elliptic* integrals. This comes from the fact that E is represented by *sums* of Abelian integrals in analogy with the imaginary part appearing in formula (5) above.

IV) THE SYSTEM OF CRYSTAL OPTICS

If $\mathcal{H}$ denotes the magnetic field, and $\mathcal{J}$ denotes the density of current, and $\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$, μ, c, ϵ_j being positive constants, then

$$\big(I_3\partial_t^2 + A(\nabla)\big)\mathcal{H} = \frac{4\pi c}{\mu}\,\mathrm{rot}(\epsilon^{-1}\mathcal{J})$$

with the symmetric matrix

$$A(\xi) = \begin{pmatrix} -d_3\xi_2^2 - d_2\xi_3^2 & d_3\xi_1\xi_2 & d_2\xi_1\xi_3 \\ d_3\xi_1\xi_2 & -d_3\xi_1^2 - d_1\xi_3^2 & d_1\xi_2\xi_3 \\ d_2\xi_1\xi_3 & d_1\xi_2\xi_3 & -d_1\xi_2^2 - d_2\xi_1^2 \end{pmatrix}$$

where we have set $d_j = \dfrac{c^2}{\mu\epsilon_j}$, $j = 1, 2, 3$. Particularly important for systems of PDOs is the determinant operator. We have

$$\det\big(I_3\partial_t^2 + A(\nabla)\big) = \partial_t^2 R(\partial)$$

with

$$R(\tau,\xi)=\tau^4-\tau^2\sum_{j=1}^{3}\xi_j^2(d_{j+1}+d_{j+2})+|\xi|^2\sum_{j=1}^{3}\xi_j^2 d_{j+1}d_{j+2}$$

(where we define $d_4=d_1,\quad d_5=d_2$). The slowness surface $X=\{\xi\in\mathbf{R}^3;R(1,\xi)=0\}$ is called "Fresnel's surface". If the positive constants d_1,d_2,d_3 are pairwise different, then X is homeomorphic to two disjoint spheres glued together at four points, which two by two are pairwise opposite and span the "optical axes". In this case, R is irreducible. If two of the d_j are equal, i.e., if $d_1=d_2=1$ and $d_3=d\neq 1$ without loss of generality, then the crystal is called "uniaxial", because in this case the optical axes coincide. Then X is made up of a sphere and of an ellipsoid touching each other at the two points on the optical axis.

Use of the matrix version of the Herglotz-Petrovsky formula yields a representation of E by Abelian integrals (cf. [OW3, 2.2.2], [KS, p. 3318])

$$E(t,x)=-\frac{Y(t)}{4\pi^2}\,\partial_t\int_{C_{t,x}}\frac{P(1,\xi)^{\mathrm{ad}}\,\mathrm{sign}\big((\partial_\tau\det P)(1,\xi)\big)}{|x_3(\partial_2\det P)(1,\xi)-x_2(\partial_3\det P)(1,\xi)|}\,|\mathrm{d}\xi_1| \tag{6}$$

where

$$C_{t,x}:=\{\xi\in\mathbf{R}^3;\det P(1,\xi)=0,t+\langle\xi,\,x\rangle=0\}\qquad(\text{for }(t,x)\in\mathbf{R}^4).$$

Unfortunately, in the case of crystal optics (i.e., $P(\partial)=\partial_t^2-A(\nabla),\quad A$ as above), an explicit evaluation (in terms of higher transcendental functions) of formula (6) has not yet been achieved. Let me describe what is known so far ([OW3, 3.4 and 4.3]):

If K denotes the support of E, then K is the dual cone of the connectivity component of $(1,0)$ in $\{(\tau,\xi)\in\mathbf{R}^4;R(\tau,\xi)\neq 0\}$. The singular support of E (which coincides with the wave front surface W) consists of the four "Hamiltonian circles" and of $\{(t,x);t>0,x/t\in X^0\}$, where X^0 is the dual surface to the slowness surface X, i.e. X^0 is the envelope of the planes $\{x\in\mathbf{R}^3;\langle\xi,x\rangle=1,\xi\in X\}$. It turns out that

$$X^0=\{x\in\mathbf{R}^3;1-\sum_{j=1}^{3}x_j^2(d_{j+1}^{-1}+d_{j+2}^{-1})+|x|^2\sum_{j=1}^{3}x_j^2/(d_{j+1}d_{j+2})=0\}$$

and hence X^0 is given by an equation analogous to that of Fresnel's surface X. The intersection of X^0 with a plane through the optical axes consists of a circle intersecting an ellipse, see Fig. 4.

The fundamental matrix E is explicitly known in the inner region J. There

$$E=\mathrm{vp}\,\frac{tY(t)}{4\pi|x|^3}\Big(I_3-\frac{3x\cdot x^T}{|x|^2}\Big)+\frac{1}{3}\big(tY(t)\otimes\delta(x)\big)I_3.$$

Furthermore, one can calculate the delta terms in E, and E is known in the uniaxial case. In this case, the circle and the ellipse in Fig. 4 touch each other and E can be expressed by delta terms and algebraic functions ([OW3, Prop. 3]). For the biaxial case, however, E is given in $K\backslash J$ by Abelian integrals over curves of genus 3, the moduli of which depend on (t,x). Up to now, there is no representation by higher transcendental functions known.

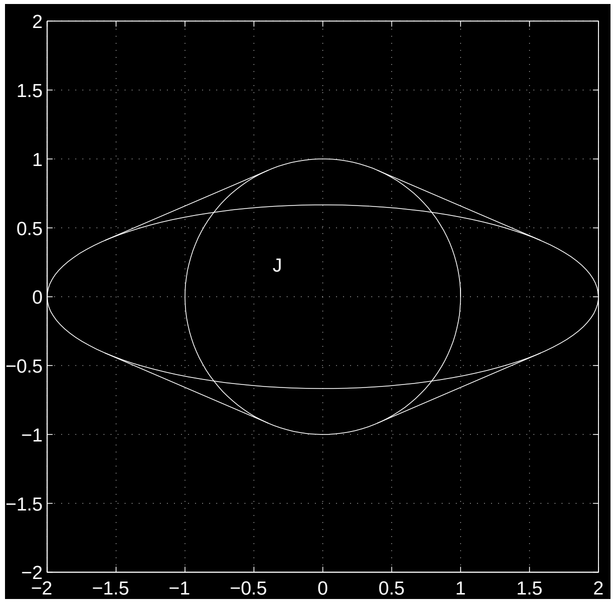

Figure 4: Section of the wave front surface of crystal optics through the optical axes

Bibliography

[A] d'Alembert, J. le R., *Recherches sur la courbe que forme une corde tendue mise en vibration*, Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin **3** (1747/1749), 214–219.

[ABG] Atiyah, M. F., R. Bott, and L. Gårding, *Lacunas for hyperbolic differential operators with constant coefficients I, II*, Acta Math. **124** (1970), 109–189; **131** (1973), 145–206.

[B] Боровиков, В. А., Фундаментальные решения линейных уравнений в частных производных с постоянными коэфициентами, Труды Моск. матем. об-ва **8** (1959), 199–257; Engl. transl.: V. A. Borovikov, *Fundamental solutions of linear partial differential equations with constant coefficients,* Amer. Math. Soc. Transl., Ser. 2, **25** (1963), 11–66.

[E] Ehrenpreis, L., *Solution of some problems of division. Part I. Division by a polynomial of derivation*, Am. J. Math. **76** (1954), 883–903.

[F] Fourier, J., *Note relative aux vibrations des surfaces élastiques et au mouvement des ondes*, Bull. Sci. Soc. Philomathique Paris (1818), 129–136.

[Fr1] Fredholm, I., *Sur les équations de l'équilibre d'un corps solide élastique*, Acta Math. **23** (1900), 1–42; Œuvres complètes: 17–58.

[Fr2] Fredholm, I., *Sur l'intégrale fondamentale d'une équation différentielle elliptique à coefficients constants*, Rend. Circ. Mat. Palermo **25** (1908), 346–351; Œuvres complètes: 117–122.

[GH] Griffiths, Ph., and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[GS] Гельфанд, И. М., и Г. Е. Шилов, Обобщённые функции и действия над ними, Физматгиз, Москва, 1958; Engl. transl.: I. M. Gel'fand and G. E. Shilov, *Generalized functions I, properties and operations*, Academic Press, New York, 1964.

[GR] Gradshteyn, I. S., and I. M. Ryzhik, *Tables of series, products, and integrals*, Harri Deutsch, Thun, 1981.

[He] Herglotz, G., *Über die Integration linearer partieller Differentialgleichungen mit konstanten Koeffizienten I–III*, Ber. Sächs. Akademie Wiss. **78** (1926), 93–126, 287–318; **80** (1928), 69–114; Ges. Schriften: 469–607.

[H1] Hörmander, L., *On the theory of general partial differential operators*, Acta Math. **94** (1955), 161–248.

[H2] Hörmander, L., *Local and global properties of fundamental solutions*, Math. Scand. **5** (1957), 27–39.

[H3] Hörmander, L., *Linear partial differential operators*, Grundlehren, 116, Springer, Berlin, 1963.

[H4] Hörmander, L., *The analysis of linear partial differential operators. Vol. I–IV*, Grundlehren, 256/7, 274/5, Springer, Berlin, 1983, 1985.

[KS] Khutoryansky, N. M., and H. Sosa, *Dynamic representation formulas and fundamental solutions for piezoelectricity*, Int. J. Solids Struct. **32** (1995), 3307–3325.

[K] Kirchhoff, G., *Zur Theorie der Lichtstrahlen*, Sitz.ber. K. Preuss. Akad. Wiss. Berlin (1882), 641–669.

[Kö] König, H., *An explicit formula for fundamental solutions of linear partial differential equations with constant coefficients*, Proc. AMS **120** (1994), 1315–1318.

[L1] Laplace, P. S., *Mémoire sur la théorie de l'anneau de Saturne*, Mém. Acad. Roy. Sci. Paris (1787/1789), 201–234; cf. also Traité de mécanique céleste II, no. 11, and Œuvres XI, 275–292.

[L2] Laplace, P. S., J. École Polytéch. cah. 15 (1809), p. 240 (quoted in Enc. Math. Wiss. Band II, 1. Teil, 2. Hälfte, p. 1198).

[Le] Leray, J., *Hyperbolic differential equations*, Institute of Advanced Study, Princeton, 1952.

[Ł] Łojasiewicz, S., *Sur le problème de la division*, Studia Math. **18** (1959), 87–136.

[Lü] Lützen, J., *The prehistory of the theory of distributions*, Springer, Berlin, 1981.

[M] Malgrange, B., *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier **6** (1955/56), 271–355.

[MTV] Meise, R., B. A. Taylor, and D. Vogt, *Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse*, Ann. Inst. Fourier **40** (1990), 619–655.

[O] Ortner, N., *Regularisierte Faltung von Distributionen. Teil 1: Zur Berechnung von Fundamentallösungen. Teil 2: Eine Tabelle von Fundamentallösungen*, ZAMP **31** (1980), 133–173.

[OW1] Ortner, N., and P. Wagner, *A short proof of the Malgrange-Ehrenpreis theorem*, Functional Analysis, Proc. of the First Int. Workshop in Trier, ed. by S. Dierolf, S. Dineen, P. Domański, de Gruyter, Berlin, 1996, pp. 343–352.

[OW2] Ortner, N., and P. Wagner, *A survey on explicit representation formulae for fundamental solutions of linear partial differential operators*, Acta Applicandae Math. **47** (1997), 101–124.

[OW3] Ortner, N., and P. Wagner, *Fundamental matrices of homogeneous hyperbolic systems. Applications to crystal optics, elastodynamics and piezoelectromagnetism*, ZAMM (J. Appl. Math. Mech.) (to appear).

[Pe] Petrovsky, I. G., *On the diffusion of waves and the lacunas for hyperbolic equations*, Mat. Sb. **17** (1945), 283–370.

[P1] Poisson, S. D., *Remarques sur une équation qui se présente dans la théorie de l'attraction des sphéroïdes*, Bulletin de la Société Philomathique de Paris **3** (1813), 388–392.

[P2] Poisson, S. D., *Sur l'intégrale de l'équation relative aux vibrations des surfaces élastiques et au mouvement des ondes*, Bulletin de la Société Philomathique de Paris (1818), 125–128.

[P3] Poisson, S. D., Mém. Acad. Sci. Paris **3** (1818), p. 131 (quoted in Enc. Math. Wiss. Band II, 1. Teil, 2. Hälfte, p. 1258).

[P4] Poisson, S. D., *Mémoire sur l'équilibre et le mouvement des corps élastiques*, Mém. Acad. Sci. Paris **8** (1829), 357–570.

[Sch] Schwartz, L., *Théorie des distributions*, Nouv. éd., Hermann, Paris, 1966.

[St] Stokes, G. G., *On the dynamical theory of diffraction*, Trans. Cambridge Phil. Soc. **9** (1849), 1–62.

[V] Volterra, V., *Sur les vibrations des corps élastiques isotropes*, Acta Math. **18** (1894), 161–232.

[W1] Wagner, P., *Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions*, Acta Math. **182** (1999), 283–300.

[W2] Wagner, P., *A fundamental solution of N. Zeilon's operator*, Math. Scand. **86** (2000), 273–287.

[W3] Wagner, P., *Zeilon's operator and lacunae*, Functional-Analytic and Complex Methods, their Interactions and Applications to Partial Differential Equations, Proc. PDE-Workshop 2001 in Graz, ed. by H. Florian, N. Ortner, F. Schnitzer, W. Tutschke, World Scientific, Singapur, 2001, pp. 33–46.

[W4] Wagner, P., *On the fundamental solutions of a class of elliptic quartic operators in dimension 3*, J. Math. Pures Appl. **81** (2002), 1191–1206.

[W5] Wagner, P., *On the fundamental solutions of a class of hyperbolic quartic operators in dimension 3*, Ann. Mat. Pura Appl. (to appear).

[Z1] Zeilon, N., *Das Fundamentalintegral der Allgemeinen Partiellen Linearen Differentialgleichung mit konstanten Koeffizienten*, Arkiv f. Mat. Astr. o. Fys. **6** (1911), 1–32.

[Z2] Zeilon, N., *Sur les intégrales fondamentales des équations à charactéristique réelle de la Physique Mathématique*, Arkiv f. Mat. Astr. o. Fys. **9** (1913/14), no. 18, 1–70.